

Derivation of Poincare Invariance from general quantum field theory

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Abstract

Starting from a very general quantum field theory we seek to derive Poincaré invariance in the limit of low energy excitations. We do *not*, of course, assume these symmetries at the outset, but rather only a very general second quantised model. Many of the degrees of freedom on which the fields depend turn out to correspond to a higher dimension. We are not yet perfectly successful. In particular, for the derivation of translational invariance, we need to assume that some background parameters, which *a priori* vary in space, can be interpreted as gravitational fields in a future extension of our model. Assuming translational invariance arises in this way, we essentially obtain quantum electrodynamics in just $3 + 1$ dimensions from our model. The only remaining flaw in the model is that the photon and the various Weyl fermions turn out to have their own separate metric tensors.

1 Introduction

One of the most shocking results of Einstein's theory of relativity, which again and again make people feel mystified, is the relativity of time or of simultaneity. With our prejudice about time, we have a natural tendency to believe in an absolute time. In the present work we shall present a model which has an absolute time without either Lorentz invariance or rotational invariance at the outset. Nevertheless, under very general assumptions, it essentially leads to the familiar Poincaré invariant quantum electrodynamics.

This work is to be considered part of our long-standing ambitious project of Random Dynamics [1], which attempts to *derive* rather than assume the laws of nature, preferably from an “extremely complicated fundamental world machinery”. That is really to say that we assume that the fundamental laws of nature, or the fundamental world machinery we could say, is so *enormously complicated* that it would be hopeless to attempt to guess what it is and write it down in detail. The best one could hope for might be to give an idea of what type it is, but it would have so enormously many unknown parameters that we would, in practice, have to just assume that these parameters were random and hope that this would not matter. In this sense, the Random Dynamics philosophy consists of taking the fundamental world machinery to be so complicated and so unspecified that, in reality, we have assumed close to nothing. Then we hope that, in spite of assuming that it almost does not matter what this world machinery is and what its parameters are, we can get out *in some limits* the laws of nature as we know them. We should consider the present work as one little link in the long chain of deductions, in principle leading from the random world machinery to the present laws – i.e. the Standard Model of elementary particle physics.

1.1 Central point of the article

With our attempts to derive “everything” the article gets long and complicated. The spirit of the article is to attempt to derive properties that are normally simply postulated and taken for granted. For example we shall spend some time discussing the origin of the rudiments of translational invariance. So let us here summarize the main points in our argument for Lorentz invariance and for the number of spatial dimensions being effectively three for the “poor physicist”, who can only use very low energies per particle compared to the fundamental (Planck) mass scale. Such a physicist can safely Taylor expand to lowest order in the momentum \vec{p}_{phys} around that value for which the available single particle states actually have their energy approach zero in the boson case, or the Fermi surface in the fermion case¹. Let us, as shall anyway be done below, “renormalise” the momentum so that we have zero energy for zero momentum, by shifting the momentum notation by an additive constant, and expand around zero momentum. We shall generally argue that the equation of motion, for the fields in momentum representation, takes the form of the energies being given by the eigenvalues of an *antisymmetric real* matrix, when we work with real fields – corresponding to thinking of the single particle wave function as being real, i.e. separated into real and imaginary parts in momentum representation. Let us also here imagine, for the illustration of our main point, that we have already managed to argue for fermions that we effectively only have to consider four real fields and for bosons only three. Let us even assume in the fermion case that – as we shall see – the only relevant part of the four by four matrix is the piece which commutes with the matrix describing the single particle Hamiltonian with zero momentum. Really we shall see in section 8 that this zero momentum Hamiltonian functions as the imaginary element i . It follows, in the fermion case, that instead of an antisymmetric real four by four matrix we can work effectively with an Hermitian two by two matrix.

¹Note that here boson and fermion purely refer to the statistics of the particles. At this stage we are not even assuming rotational invariance, let alone the spin-statistics theorem.

That is to say we shall effectively argue for field equations of the form

$$\dot{\xi}_i(\vec{p}) = \sum_k A_{ik} \xi_k(\vec{p}) \quad (1)$$

where the ξ_i refer to the fermion or boson fields, with the index i running *a priori* over all the many at first existing Hermitian or real fields, but effectively taken to run only over 3 for bosons and 4 for fermions. Furthermore, in the fermion case, we may even shift to having instead only two i values but, to make up for this, the fields become complex. The matrix elements of the above mentioned antisymmetric real matrix are here denoted A_{ik} and they are functions of the momentum in a momentum conserving theory. (Here momentum conservation is just put in for pedagogical simplicity; in much of the present article we seek to work without assuming momentum conservation.)

One of our main achievements is the prediction of the dimensionality of space to be three – with the one time dimension in addition already assumed at the outset. This prediction arises from the fact that, in the boson case, there are just three linearly independent 3 x 3 real antisymmetric matrices and, similarly in the fermion case, three linearly independent 2 x 2 Hermitian matrices in addition to the unit matrix.

For instance, by an appropriate basis (coordinate) choice in momentum space, the Taylor expansion of the 3 x 3 real antisymmetric matrix for small momenta can be expressed in the form:

$$A = \left(\begin{array}{c|c|c} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{array} \right) \quad (2)$$

Although the boson fields considered here have three components which, as already stressed, are real (or Hermitian when second quantized), we have a trick one could say of getting *both a magnetic and an electric field* out from them. We shall see that, in the position or \vec{x} -representation, it corresponds to a splitting into real and imaginary parts. This trick involves letting the universe become an orbifold, or equivalently of assuming the existence of a point around which the world is in a parity symmetric *state*.

The crucial observation in our work is that, by this sort of most general low momentum Taylor expansion, we actually happen to get well-known *Lorentz invariant* equations of motion for the fields, such as the Maxwell equations for the boson case and the Weyl equation for the fermion case. In fact if we denote the three components of the real boson field in momentum representation as a three vector $\vec{\phi}(\vec{p})$, we may interpret it in terms of momentum representation electric and magnetic fields by

$$\vec{\phi} = \vec{B} + i\vec{E}. \quad (3)$$

This equation may look a bit strange at first, when we stress so much the reality of $\vec{\phi}$, but one should remember that the Fourier transformed \vec{B} and \vec{E} are *not* real even if the \vec{x} -representation electric and magnetic fields are real. With the funny orbifold picture we argue should appear in our model, we would rather get for example that \vec{E} is purely imaginary.

The present article is built around this derivation of the already so well-known and empirically so successful equations of motion by a Taylor expansion in momentum, starting from an exceedingly general field theory. We first describe how we are led, with an appropriate interpretation in terms of background gravitational fields, to a kind of formal translational invariance. Then we give a speculative argument for the number of relevant field components at low energy. After deriving the free Maxwell and Weyl equations, we allow for interactions. By imposing restrictions from the required self-consistency of the equations of motion with interactions included, we get very close to quantum electrodynamics in a background gravitational field. However it must be

admitted there are still small flaws, the major one of which is that the electron and photon get different metric tensors. This may mean we have two gravity fields so far, but we hope to progress further with this approach.

1.2 Plan of the article

In the following section 2 we shall take as our starting point a very general quantum field theory, so general that we do *not* at first assume the usual physical principles such as translational invariance and Lorentz (including rotational) invariance. Indeed we do not really even assume that there is a space; rather we assume that the fields depend on a number of parameters, which are initially taken to be greater than 3. One of our main results is precisely to derive the existence of a space effectively having just 3 dimensions. For simplicity and to be specific, we do though assume an absolute time, which we hope is not so important in our model, and time translational invariance. In section 3 we seek to argue that the two (and higher) point functions in the very general quantum field theory must effectively have the delta-function structure of just the same type as in momentum conserving theories. However this does not yet correspond to momentum conservation, but only to the absence of potential terms in the integration kernels of the theory which do not have this singularity structure. We give some speculative hopes and an explanation of why we cannot expect, at least from general arguments, to explain away the possibility of some background field such as a background gravitational field. In section 4 we then write useful expressions for the two point kernels with the suggested singularities and study their symmetry and Hermiticity properties, especially formulated in terms of commutator functions, anti-commutator functions and single particle Hamiltonian functions extended to even be considered as functions on the single particle phase space. Actually an important point is that, due to our assumption that we should work with real fields (or rather Hermitian fields in the second quantized language), the phase space gets folded up as an orbifold and we actually only get a “half phase space” – or really halved down once for each dimension. The most basic idea is to Taylor expand the four functions, describing the two point kernels and defined in section 4, on phase space in the very small range of momenta accessible, for practical purposes, to a physicist working only with low energies compared to the fundamental scale. In the next section 5 we choose the normalization of the fields, so as to simplify the Hamiltonian in the case of bosons but the anti-commutator in the case of fermions. It turns out that, in both cases, we end up with having to seek eigenvalues for antisymmetric (and apart from an over all factor i) real matrices.

In section 6 we present some effects for the very general type of system we really consider: a system that is second quantized with bosons and/or fermions and some degrees of freedom that can adjust to lower the energy *under the influence of the “zero point energy”* of the second quantized degrees of freedom. For fermions such an effect is known – by chemists – under the name of Homolumo-gap. For the bosons we talk about the “analogue of the Homolumo gap effect” and the effect is that the system will adjust itself so that very many boson frequencies (= boson single particle energies) become exactly zero. The exactness of the effect is connected with the fact that the zero point energy has a “numerical value singularity”.

Our derivations of Lorentz (and rotational) invariance in the free boson and free fermion cases are put into sections 7 and 8 respectively. For these derivations the zero-point energy effects are quite important and, to be honest, we may even use these effects in a bit too optimistic way.

In section 9 we call attention to the fact that, in these type of models, one tends to get a charge conservation symmetry as an accidental symmetry. This fact is actually of importance in avoiding problems with the CPT-theorem, due to level repulsion of the CPT-degenerate states which otherwise would be expected. In a generic theory – as we work with – one should not expect degeneracies to occur without reason.

In section 10 we study the very general interaction that might occur in our general field theory

between the boson fields, which we now interpret as the Maxwell fields, and the fermion fields, which we interpret as the Weyl field for a supposedly charged particle. We basically find that, in the general Weyl equation, bosonic degrees of freedom appear, which formally look like the gauge potential A . Then we study the dominant commutator of this A -field with the Maxwell fields, which we already obtained in the free approximation. A major point, in arguing for the couplings we shall arrive at, is to require consistency of the equations of motion with the interaction included. In the last section 13 we present the conclusion and some outlook.

2 A very general quantum field theory

We consider here a very generalized field theory model, in the sense that *a priori* we do not even have an interpretation of the fields, $\phi_i(\vec{x}_{pre})$ say, as being functions of a spatial point \vec{x}_{pre} . Instead, we rather think of this ordered set of parameters $\vec{x}_{pre} = (x_{pre}^1, x_{pre}^2, \dots, x_{pre}^d)$ as only “some parameters on which the fields $\phi_i(\vec{x}_{pre})$ depend”. For example, later they might turn out to be interpreted as momentum rather than position and/or the true position or momentum may turn out to be complicated functions of these \vec{x}_{pre} -parameters and their conjugates \vec{p}_{pre} . One might think of these primitive parameters \vec{x}_{pre} and their conjugate single particle variables \vec{p}_{pre} as kind of pre-position and pre-momentum. In the following we shall in practice only consider a small region of momentum space (i.e. the low energy region accessible to experiment) and Taylor expand in momentum e.g. terms in the Hamiltonian, energies of particles, and the above-mentioned complicated functions, which relate the primitive variables to the single particle dynamical variables eventually identified with the observed position and momentum.

In our generalized field theory, the field operators $\phi_i(\vec{x}_{pre})$, which are strictly speaking operator-valued distributions, act on the Hilbert space of the whole underlying model. In addition to the considered fields, we imagine that the full world machinery of our model contains other degrees of freedom, which we shall refer to as the “rest” degrees of freedom. So the full Hilbert space is the Cartesian product of a Hilbert space analogous to the Fock space in standard quantum field theory and some further “rest” Hilbert space.

In essentially the usual way, we shall consider continuous and differentiable functions/distributions of the single particle pre-position and pre-momentum variables \vec{x}_{pre} and \vec{p}_{pre} . By this we mean that the kernels, (anti-) commutators etc. appearing in the general theory are assumed to be expressed in terms of genuine functions and the most suggestive distributions, i.e. the δ -functions. It should be stressed again here that, in the spirit of random dynamics, we are considering a random and *a priori* complicated model or world machinery. In particular, we do not put the usual translational or Lorentz symmetries into the model. Rather our hope is to derive Poincaré invariance, as an effective or accidental symmetry for a low energy observer.

For simplicity, we shall assume that the model has an absolute time t and a usual quantum mechanical development equation:

$$i \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \quad (4)$$

where $|\psi\rangle$ is a ket belonging to the Hilbert space. The very general Hamiltonian is taken to be of the smooth form

$$H = \sum_{n=0}^{\infty} \int K_{i_1 i_2 \dots i_n}(\text{rest}; \vec{x}_1^{pre}, \vec{x}_2^{pre}, \dots, \vec{x}_n^{pre}) \phi_{i_1}(\vec{x}_1^{pre}) \phi_{i_2}(\vec{x}_2^{pre}) \dots \phi_{i_n}(\vec{x}_n^{pre}) d^d \vec{x}_1^{pre} d^d \vec{x}_2^{pre} \dots d^d \vec{x}_n^{pre} \quad (5)$$

as far as the fields are concerned. Here “rest” in the kernel K indicates that K would normally also be an operator in the “rest” Hilbert space. Note that here and in the following, we use the subscript *pre* and the superscript *pre* interchangeably.

3 Derivation of translational invariance and locality, at first only singularity structure

We now want to discuss how momentum conservation might arise in such a general quantum field theory. Since, as is well-known, momentum conserving kernels have the singular form of a delta-function, say $\delta(\vec{p}_{pre} - \vec{p}'_{pre})$ for the two-field case, we need to discuss the singularity structure of the kernels $K_{i_1 i_2 \dots i_n}(rest; \vec{x}_1^{pre}, \vec{x}_2^{pre}, \dots, \vec{x}_n^{pre})$ in equation (5) in order to at least argue for the right singularity structure. *A priori* one could think of these kernels as being smooth functions or smooth functions multiplied by some products of δ -functions. If the kernel had no δ -function with respect to \vec{x}_1^{pre} say, then you would only get a non-zero overlap with wave functions for a limited range in the conjugate variable \vec{p}_1^{pre} . However, it would then turn out that one of the terms in equation (5) only depends on one of the fields ϕ_{i_1} for a small region in the single particle $(\vec{x}_1^{pre}, \vec{p}_1^{pre})$ phase space.

As this example of a smooth kernel shows, one can very easily risk to get a kernel that projects away a huge number of states approximately.

One way of presenting the problem of avoiding the projection to almost zero of a huge number of states is to imagine approaching the continuous kernel from a lattice; say we replace the continuous – real – variables or vectors \vec{x}_i^{pre} by discretized ones with some lattice distance a between the points on the coordinate axes which are actually included in the step considered. Then, of course, the idea is to gradually make the lattice distance smaller and smaller and at the end let $a \rightarrow 0$. For example, we could think of this limit being taken by successively diminishing the lattice constant by a factor 2, but really the details of how one makes the lattice more and more dense should not matter for our argument.

3.1 The no-go argument for smooth kernels without decoupling

In order to show the decoupling for the great majority of states when one has a smooth kernel, let us think of the regularization of the operator produced by convolution with the kernel $K_{i_1 i_2}(rest; \vec{x}_1^{pre}, \vec{x}_2^{pre})$ as a limit of a series of lattice replacement matrices. These correspond to replacing the wave functions defined on the vector space of \vec{x}_i^{pre} -values by “columns” of numbers, with elements being the wave function values at the sites in a lattice introduced into this vector space; the kernel itself is replaced by a matrix with rows and columns marked by the lattice points. Denoting this matrix associated with the kernel by \mathbf{K} , the result of the convolution would be given as the limit of the lattice constant a going to zero of the matrix multiplication $\mathbf{K}\phi$. The crux of the argument now is that, as we go along diminishing a and thereby increasing the order of the matrix \mathbf{K} , we get more and more eigenvectors and eigenvalues with the added eigenvalues becoming smaller and smaller! We may estimate the order of magnitude of the size of the added eigenvalues, by estimating how the determinant varies as we add more and more lattice points. Imagine that we simply add one more lattice point. Then we add one row and one column to the matrix representing the convolution \mathbf{K} . So its determinant is modified by a factor given by what we may write in the added row, after having reduced the numbers in this row by adding to it linear combinations of the other (old) rows. Now, however, we can almost reproduce the numbers in the new row corresponding to the new lattice point, by Taylor expansion interpolations from the “old” lattice points, because of the assumed smoothness of the kernel we study. The interpolation estimate of a new row is a linear combination of the other rows. As the lattice gets more and more dense, the order to which we can reproduce the Taylor series accurately gets higher and higher, say up to a term involving the p th derivative. Then the size of the deviation of the true new row from its interpolation estimate becomes of the order a^p times the typical size of the kernel values. This means that, as we add more and more lattice points, the determinant gets suppressed by a factor of the order a^p for each extra lattice point. Since the determinant is the product of all

the eigenvalues, this in turn means that the added eigenvalues must be of the order a^p . That in turn means that by far the majority of the eigenvalues of the operator, corresponding to the convolution with the assumed smooth kernel, are extremely small, since the factors a^p go to zero very fast in the limit of the lattice becoming more and more dense. We can also express this result by saying that by far the majority of the wave functions, on which the smooth kernel acts by convolution, are transformed into extremely small (essentially vanishing) wave functions. Thus we can consider that most of the state space is actually *decoupled* w.r.t. such a kernel.

It is really rather easy to understand what is going on, by considering the various smooth functions $K_{i_1 i_2 \dots i_n}(\text{rest}; \vec{x}_{10}^{pre}, \vec{x}_{20}^{pre}, \dots, \vec{x}_{(p-1)0}^{pre}, \vec{x}_p^{pre}, \vec{x}_{(p+1)0}^{pre}, \dots, \vec{x}_{n0}^{pre})$, obtained out of a kernel $K_{i_1 i_2 \dots i_n}$ (which could now depend on several variables) when all the vector-variables are kept fixed, $\vec{x}_i = \vec{x}_{i0}$, except for one \vec{x}_p . The resulting smooth functions are approximately linearly dependent upon each other due to interpolation. Therefore all the wave functions $\phi_{i_p}(\vec{x}_p^{pre})$ “orthogonal” to the little subspace spanned by these functions have essentially zero eigenvalues or, equivalently, decouple. For by far the majority of wave functions we can thus ignore the smooth part of the kernels, without in practice causing much disturbance.

3.2 Next level of smoothness

So now we have to weaken our smoothness assumption concerning the kernels. Here we have to make a guess, based on aesthetics and general experience, as to what type of functions one meets in physics next to the obvious and usual smooth functions. We consider it fair to say that, at the next level of smoothness, the types of function which often occur physically are the Dirac delta-function and its derivatives. In order not to come into the problem with the smooth kernels, it is necessary that there is a delta-function behaviour w.r.t. any one of the sets of variables on which the kernel K depends. Let us take the delta-function peak to be at the geometrical site for which a certain function $\vec{f}(\vec{x}_1^{pre}, \dots, \vec{x}_n^{pre})$ is zero. Then the “next type of function in the series with different degrees of smoothness” is obtained as the delta-function $\delta(\vec{f}(\vec{x}_1^{pre}, \dots, \vec{x}_n^{pre}))$ differentiated various numbers of times w.r.t. the various variables and weighted with various genuinely smooth functions. That is to say we can write the kernel

$$K = \sum_j F_j(\vec{x}_1^{pre}, \dots, \vec{x}_n^{pre}) \delta^{(j)}(\vec{f}(\vec{x}_1^{pre}, \dots, \vec{x}_n^{pre})) \quad (6)$$

where j is an integer that runs from 0 to ∞ , and the symbol (j) means the j th derivative.

One could easily get several terms of the form just described. But if the “poor physicist” only sees the phase space locally, he will only see one of the delta-functions and it will not be important if there are more of them.

Now let us note that the derivatives on the delta-functions of \vec{f} could be produced by using derivatives w.r.t. any of the variables \vec{x}_i^{pre} . In turn such derivatives are, apart from a factor i , the operators corresponding to the conjugate variables of the \vec{x}_i^{pre} s. We can thus replace the whole construction of derivatives *and* the F_j factors in front, by the action of an operator corresponding to the quantum mechanical representation of some classical function (defined on the phase space for all the variables \vec{x}_i and their conjugates). That is to say that we can write the general expression in the form:

$$K = F(\vec{x}_1^{pre}, \dots, \vec{x}_n^{pre}, \vec{p}_1^{pre}, \dots, \vec{p}_n^{pre}) \delta(\vec{f}(\vec{x}_1^{pre}, \dots, \vec{x}_n^{pre})). \quad (7)$$

At first we may see the seed for a conservation law in the delta-function, since we could attempt to identify the \vec{f} as the physical momentum. However, as we see, there are derivatives of the delta-functions or, in the last equation, the dependence on all the phase space variables, which would normally spoil the translational invariance needed for momentum conservation. Anyway we may

perform the just mentioned step in making it look like a momentum conserving kernel, by formally writing \vec{f} as a sum of “physical momenta” $\vec{p}_{phys,i}$ in the region accessible to the poor physicist

$$\vec{f} = \sum_i \vec{p}_{phys,i}. \quad (8)$$

But, at first, we could not expect to get this simple form (8), if the fields depending on the different \vec{x}_i^{pre} and corresponding momenta \vec{p}_i^{pre} are defined on the same space of parameters ($\vec{x}_i^{pre}, \vec{p}_i^{pre}$). Rather there would be an additive constant vector term in equation (8). If the parameter spaces (or phase spaces) are different, however, in the sense that the different parameters \vec{x}_i^{pre} are *a priori* of a different nature and setting them equal would not be physically meaningful, we can define the $\vec{p}_{phys,i}$ with such a constant absorbed so as to really get equation (8). The crux of the matter is that the identification of the same point for two different fields is only established by the use of the singularity of the kernel itself. Now, in the “poor physicist” philosophy, the problematic constant vector will almost always be huge on the “poor physicist” scale. So the regions in phase space, or momentum space, connected by the delta-function will be so far apart that we can easily treat them as belonging effectively to different spaces. According to this philosophy, we should now really think of the typical situation as being that the different field components ϕ_i, ϕ_k are *a priori* defined on different small spots in phase or momentum space, which are identified as relevant by the delta-functions in the kernels. In this philosophy the field, with a given index i , would not in practice come to multiply itself (because if it did you would give it a new name) in say the Hamiltonian (5). However once we begin to make linear transformations on the fields with the same physical momenta into each other, such diagonal terms can easily appear.

In this way we get that the delta-function looks momentum conserving, but we also have the factor F in the kernel which depends on the conjugate variables to $\vec{p}_{phys,i}$. So the kernel does not conserve momentum. It is rather locally of a momentum conserving character, but since it depends on position – here we talk about the physical position – it is *not* translational invariant. However we speculate later that this \vec{x} dependence could be “dynamical” (i.e. depend on the “rest” variables) and somehow be adjusted at low energy, or under some cosmological conditions, to become effectively translational invariant after all.

3.3 Commutators and anti-commutators

In an analogous spirit we shall now argue for the singularity structures for the commutators and anti-commutators.

Let us in fact consider the commutator or anti-commutator (as the statistics for the fields will require to be considered) of two fields $\phi_i(\vec{x}_{pre})$ and $\phi_j(\vec{x}'_{pre})$:

$$[\phi_i(\vec{x}_{pre}), \phi_j(\vec{x}'_{pre})]_{\pm} = f_{ij}(\vec{x}_{pre}, \vec{x}'_{pre}) + \text{“field operators”} \quad (9)$$

where we shall assume that the c-number term dominates (this is like a free approximation would suggest). From this commutator or anti-commutator we can of course deduce, using the Fourier transformation formulae

$$\phi_i(\vec{x}_{pre}) = \int \tilde{\phi}_i(\vec{p}_{pre}) \exp(i\vec{p}_{pre} \cdot \vec{x}_{pre}) \frac{d^d \vec{p}_{pre}}{(2\pi)^d}, \quad (10)$$

the commutator or anti-commutator of the pre- \vec{p} -space representation fields

$$[\phi_i(\vec{p}_{pre}), \phi_j(\vec{p}'_{pre})]_{\pm} = \tilde{f}_{ij}(\vec{p}_{pre}, \vec{p}'_{pre}) + \text{“field operators”} \quad (11)$$

where

$$\tilde{f}_{ij}(\vec{p}_{pre}, \vec{p}'_{pre}) = \int \exp(-i\vec{p}_{pre} \cdot \vec{x}_{pre} - i\vec{p}'_{pre} \cdot \vec{x}'_{pre}) f_{ij}(\vec{x}_{pre}, \vec{x}'_{pre}) d^d \vec{x}_{pre} d^d \vec{x}'_{pre}. \quad (12)$$

In order to argue for the singularity structure of f_{ij} or \tilde{f}_{ij} , we require that weighted integrals over the field $\phi_i(\vec{x}_{pre})$ taken to be a rapidly varying function, $\int \phi_i(\vec{x}_{pre}) f_{ij}(\vec{x}_{pre}, \vec{x}'_{pre}) d^d \vec{x}_{pre}$, should not decouple in huge amounts, otherwise the field could be said, in the above defined sense, to be “effectively decoupled”. Therefore we are *not* allowed to choose the functions f_{ij} and \tilde{f}_{ij} in formulae (9) and (11) to be so smooth as to have Taylor expansions in the variables $\vec{p}_{pre}, \vec{p}'_{pre}, \vec{x}_{pre}, \vec{x}'_{pre}$. Rather, as we saw above, we need at least a delta-function singularity as a function of (at least) one linear combination (locally) of these variables. It is suggested that, as a function of this combination variable, the behaviour should correspond to a linear combination of the delta-function and its derivatives with smooth coefficients. Such a form can in turn be rewritten as an operator corresponding to a classical function in phase space, which we may call F_{ij} (both \vec{x} 's and \vec{p} 's). Indeed, by an appropriate definition of physical momenta \vec{p}_{phys} and \vec{p}'_{phys} , we may even write the delta-function as $\delta(\vec{p}_{phys} - \vec{p}'_{phys})$. We thus manage to write a commutator or anti-commutator as

$$[\tilde{\phi}_i(\vec{p}_{phys}), \tilde{\phi}_j(\vec{p}'_{phys})]_{\pm} = F_{ij}(\vec{p}_{phys}, \vec{x}_{phys}) \delta(\vec{p}_{phys} - \vec{p}'_{phys}) + \dots \quad (13)$$

Again we can really only adjust the delta-function to be of this form, provided we are allowed to redefine independently the \vec{p}_{phys} and \vec{p}'_{phys} . We can do so if, in reality, the two fields $\tilde{\phi}_i(\vec{p}_{phys})$ and $\tilde{\phi}_j(\vec{p}'_{phys})$ are defined on different parameter spaces (or phase spaces), or if we can take them to be effectively on different spaces. This again means that we imagine the fields relevant for the “poor physicist” to be really defined on small spots here and there in the fundamental phase space. But it should be stressed that the coefficient function F_{ij} is now a function of both momentum *and* position. There is thus at this stage *no momentum conservation* yet.

It would also have been quite too much, if we could indeed have derived momentum conservation properly just from the conditions that there be no effective decoupling of a huge number of single particle states. In fact the usual commutators or anti-commutators modified by some, say, gravitational field – which due to a normalization rule for the fields has been moved from the Hamiltonian into the commutator as is done in section 5 – should have no such decoupling problem. It should therefore not be possible to exclude background fields in such a way. We only got here – what is already not so trivial – the usual form with a delta-function for the commutators, but translational invariance would require some reason why possible fields that could modify them have adjusted themselves to give a translational invariant background. We shall see later that it is suggestive to speculate that these coefficient functions F_{ij} are dynamical and, thus, make the question of their translational invariance much of the same type of logic as to why our space-time is flat in general relativity.

4 General field theory model for free theories

We shall formulate the model in terms of fields that are essentially real or obey some Hermiticity conditions, which mean that we can treat the bosonic and fermionic fields $\phi(\vec{p}_{pre})$ and $\psi(\vec{p}_{pre})$ respectively as Hermitian fields:

$$\psi^\dagger(\vec{p}_{pre}) = \psi(\vec{p}_{pre}), \quad (14)$$

$$\phi^\dagger(\vec{p}_{pre}) = \phi(\vec{p}_{pre}). \quad (15)$$

However the \vec{p}_{pre} may not at the end be the true momentum but rather what we could call pre-momentum. In any case, one can always split up a non-Hermitian field into its Hermitian and anti-Hermitian parts. This is done since, in the spirit of the random dynamics project, we do not want to assume any charge conservation law from the outset. As we shall see in section 9, however, we have a mechanism that at least would easily produce a charge conservation if we in any way

should get momentum conservation. The real reason we need this Hermiticity of the fields as a function of the pre-momentum is to get some restrictions on the matrix of commutators in the boson case.

At the first stage in the development of our model, it is assumed that we *only work to the free field approximation* and thus the Hamiltonian is taken to be bilinear in the Hermitian fields $\psi(\vec{p}_{pre})$ and $\phi(\vec{p}_{pre})$. Also possible terms proportional to dynamical fields in the commutators or anti-commutators will be ignored in the same approximation. In conventional formulations one usually normalizes the fields to make the commutators and anti-commutators simple and trivial. However, since we shall here only follow this conventional procedure for the fermion fields but not for the bosons, it should be understood that to ignore terms in the commutator (or anti-commutator) is also a free theory approximation in our notation.

Let us first work in the pre-momentum representation, in which the fields are Hermitian, and let us write down the free approximation expressions for the Hamiltonian, the anti-commutators and commutators for both fermions and bosons.

Our Hamiltonians take the following forms:

$$H_F = \frac{i}{2} \int d\vec{p}_{pre} d\vec{p}'_{pre} \sum_{i,j} \psi_i(\vec{p}_{pre}) \psi_j(\vec{p}'_{pre}) H_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre}) \quad (16)$$

and

$$H_B = \frac{1}{2} \int d\vec{p}_{pre} d\vec{p}'_{pre} \sum_{i,j} \phi_i(\vec{p}_{pre}) \phi_j(\vec{p}'_{pre}) H_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre}) \quad (17)$$

for fermions and bosons respectively. Note we are using the convention that, under the complex conjugation of expressions like H_F , the order of fermionic quantities is reversed; so we need the factor of i in equation (16) in order to make $H_{ij}^{(F)}$ real. The anti-commutators and commutators take the forms:

$$\{\psi_i(\vec{p}_{pre}), \psi_j(\vec{p}'_{pre})\} = C_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre}) \quad (18)$$

and

$$[\phi_i(\vec{p}_{pre}), \phi_j(\vec{p}'_{pre})] = iC_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre}) \quad (19)$$

for fermions and bosons respectively. According to our arguments above, the functions $C_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre})$, $C_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre})$, $H_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre})$ and $H_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre})$ are singular as delta-functions or derivatives thereof so that, for example, we have the expansions

$$C_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre}) = \sum_L C_{Lij}^{(B)}(\vec{p}_{pre}) \delta^{(L)}(\vec{p}_{pre} - \vec{p}'_{pre}) = C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) \delta(\vec{p}_{pre} - \vec{p}'_{pre}) \quad (20)$$

and

$$H_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre}) = \sum_L H_{Lij}^{(F)}(\vec{p}_{pre}) \delta^{(L)}(\vec{p}_{pre} - \vec{p}'_{pre}) = H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) \delta(\vec{p}_{pre} - \vec{p}'_{pre}). \quad (21)$$

Here the summation over the symbol L is to be understood as the summation over a set of non-negative integers, denoting the numbers of differentiations on the delta-function to be performed in the various components of the vectorial variable \vec{p}_{pre} or \vec{p}'_{pre} . In the last item of these two expansions, the symbols $C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre})$ and $H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre})$ are now to be understood as quantum mechanical operators; so that \vec{x}_{pre} is really here interpreted as a differential operator w.r.t. \vec{p}_{pre} . Strictly speaking they should thus have had different symbols to $C_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre})$ and $H_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre})$, which are rather the kernels corresponding to the operators $C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre})$ and

$H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre})$. These operators C^B , C^F , H^B and H^F are quantised versions of their classical approximations, which are of course taken as smooth functions over phase space.

It is obvious that we can – for the H 's – or must – for the C 's – choose these matrices so that they obey the matrix symmetry relations:

$$H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) = -H_{ji}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}), \quad (22)$$

$$H_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) = H_{ji}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}), \quad (23)$$

$$C_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) = C_{ji}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) \quad \text{and} \quad (24)$$

$$C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) = -C_{ji}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}). \quad (25)$$

It is also obvious that, according to the equations (18) and (19), for each single pair of values (i, j) the second quantized operators $C_{ij}^{(B)}(\vec{p}_{pre}, \vec{p}'_{pre})$ and $C_{ij}^{(F)}(\vec{p}_{pre}, \vec{p}'_{pre})$ must be Hermitian, meaning real since they are c-numbers to leading order. In turn this means that the single quantized operators $C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre})$ and $C_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre})$ associated with them by, for example, equation (20) must be Hermitian. The hermiticity properties of the analogous operators $H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre})$ and $H_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre})$ given, for example by equation (21), are inherited from the expansion of the Hamiltonians as in (17). Since the second quantized fields $\phi_i(\vec{p}_{pre})$ and $\psi_i(\vec{p}_{pre})$ are assumed to be Hermitian (real in the first quantized sense), there would be no reason to introduce anti-Hermitian terms which just cancel out in the Hermitian Hamiltonian. We can therefore just take the operators $H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre})$ and $H_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre})$ associated with the matrices in (16) and (17) to be Hermitian (first quantized wise) for each pair of indices (i, j) .

Even in addition to the above relations, we have the possibility of choosing the various functions (H 's and C 's) to be invariant under the operation with the anti-linear first quantized operator PT_{pre} , defined as simple complex conjugation of the wave function in the \vec{p}_{pre} representation. The wave functions, which we are effectively having for the single particle description, are in fact real wave functions and thus invariant under this operation. We can, so to speak, just as well average every operator under this operation with PT_{pre} anyway. For operators composed from \vec{p}_{pre} and \vec{x}_{pre} , such as the H 's and C 's, this requirement simply means symmetry under the sign shift of \vec{x}_{pre} , i.e. $\vec{x}_{pre} \rightarrow -\vec{x}_{pre}$. We can also compose this operation with the previous ones and obtain equations, which could alternatively be seen as a set of equations needed to ensure the Hermiticity of the total Hamiltonian:

$$H_{ij}^{(F)\dagger}(\vec{p}_{pre}, \vec{x}_{pre}) = H_{ij}^{(F)}(\vec{p}_{pre}, -\vec{x}_{pre}), \quad (26)$$

$$H_{ij}^{(B)\dagger}(\vec{p}_{pre}, \vec{x}_{pre}) = H_{ij}^{(B)}(\vec{p}_{pre}, -\vec{x}_{pre}), \quad (27)$$

$$C_{ij}^{(F)\dagger}(\vec{p}_{pre}, \vec{x}_{pre}) = C_{ij}^{(F)}(\vec{p}_{pre}, -\vec{x}_{pre}) \quad \text{and} \quad (28)$$

$$C_{ij}^{(B)\dagger}(\vec{p}_{pre}, \vec{x}_{pre}) = C_{ij}^{(B)}(\vec{p}_{pre}, -\vec{x}_{pre}). \quad (29)$$

Notice the minus signs on the \vec{x}_{pre} arguments on the right hand side of these equations. They come about because – once the symmetry or antisymmetry under the permutation of the indices i, j takes care of the anti-commutation in the (F) case – the expressions like (20,21) must be Hermitian, or real if we take them as c-number functions of the two pre-momenta $(\vec{p}_{pre}, \vec{p}'_{pre})$. So simple derivatives, without any extra i coefficients, are what is needed in the terms corresponding, for example, to $H_{Lij}^{(F)}(\vec{p}_{pre})$ and $C_{Lij}^{(B)}(\vec{p}_{pre})$ in the sum over the different levels of differentiation L . Thus, when we replace the derivatives by the operator \vec{x}_{pre} , we insert in reality an extra i for which we must compensate by replacing

$$\vec{x}_{pre} \longrightarrow -\vec{x}_{pre} \quad (30)$$

in the Hermiticity formulae.

We summarize the relations for the operators H 's and C 's in the following way:

From the general considerations of the singularities we find that the second quantized Hamiltonians take the form

$$H_F = \frac{i}{2} \int d\vec{p}_{pre} \sum_{i,j} \psi_i(\vec{p}_{pre}) H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) \psi_j(\vec{p}_{pre}), \quad (31)$$

$$H_B = \frac{1}{2} \int d\vec{p}_{pre} \sum_{i,j} \psi_i(\vec{p}_{pre}) H_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) \psi_j(\vec{p}_{pre}), \quad (32)$$

by combining equations (16,17) with (21) and an analogous equation for $H_{ij}^{(B)}$. Similarly the commutators of the second quantized fields are obtained by inserting equation (20) and its analogue, with B replaced by F , into equations (18,19):

$$\{\psi_i(\vec{p}_{pre}), \psi_j(\vec{p}'_{pre})\} = C_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) \delta(\vec{p}_{pre} - \vec{p}'_{pre}) \quad (33)$$

$$[\phi_i(\vec{p}_{pre}), \phi_j(\vec{p}'_{pre})] = i C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) \delta(\vec{p}_{pre} - \vec{p}'_{pre}). \quad (34)$$

Then the operators acting on the first quantized Hilbert space² can be thought upon classically as functions on the phase space (as discussed above, they are taken to be smooth functions of the operators \vec{p}_{pre} and the conjugate differential operators $\vec{x}_{pre} = i \frac{\partial}{\partial \vec{p}_{pre}}$) obeying the following symmetry properties:

1. All the matrix elements are real over the whole of phase space.
2. The matrices $C^{(B)}$ and $H^{(F)}$ are antisymmetric, while the two other matrices $C^{(F)}$ and $H^{(B)}$ are symmetric all over phase space.
3. All of the four matrices take as functions over phase space the *same* values for $(\vec{p}_{pre}, \vec{x}_{pre})$ and for $(\vec{p}_{pre}, -\vec{x}_{pre})$, i.e. we have the relations

$$H_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) = H_{ij}^{(B)}(\vec{p}_{pre}, -\vec{x}_{pre}); \quad H_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) = H_{ij}^{(F)}(\vec{p}_{pre}, -\vec{x}_{pre}); \quad (35)$$

$$C_{ij}^{(B)}(\vec{p}_{pre}, \vec{x}_{pre}) = C_{ij}^{(B)}(\vec{p}_{pre}, -\vec{x}_{pre}); \quad C_{ij}^{(F)}(\vec{p}_{pre}, \vec{x}_{pre}) = C_{ij}^{(F)}(\vec{p}_{pre}, -\vec{x}_{pre}). \quad (36)$$

4.1 Canonical Transformations

We do not want to assume – and perhaps are forced not to assume if we want to get momentum conservation – that the pre-momentum \vec{p}_{pre} is necessarily to be interpreted as the true physical momentum. So we should like to allow for the possibility that the concepts of momentum and position have been canonically transformed relative to the pre-momentum and pre-position (for the single particles). We should therefore like to contemplate what happens to the (anti-)symmetry relations (22-25) and the Hermiticity relations (26-29) under canonical transformations of the $(\vec{p}_{pre}, \vec{x}_{pre})$, corresponding to unitary transformations in the single particle space. In fact we want to deduce that the number of restrictions is not changed, although their form gets more complicated.

Thinking of a classical approximation to the four functions $H_{ij}^{(F)}$, $H_{ij}^{(B)}$, $C_{ij}^{(F)}$, $C_{ij}^{(B)}$, it should be obvious that, under a canonical transformation to new variables (\vec{p}, \vec{x}) in place of the “fundamental” ones $(\vec{p}_{pre}, \vec{x}_{pre})$, we transform the equations (22-29) into the same ones just expressed

²Here the Hilbert space is of the slightly unusual type of being built on real wave functions only in the \vec{p} -representation.

in terms of the new variables. So equations (22-25) come to look quite the same, except that now the variables are the new ones (\vec{p}, \vec{x}) instead of the old ones. In order to formulate the equations (26-29) in terms of the physical coordinates, we need to give a name to the point into which the canonical transformation brings the reflected $(\vec{p}_{pre}, -\vec{x}_{pre})$ point of phase space. Let us call it $r(\vec{p}, \vec{x})$, then equations (26-29) take the form

$$H_{ij}^{(F)\dagger}(\vec{p}, \vec{x}) = H_{ij}^{(F)}(r(\vec{p}, \vec{x})), \quad (37)$$

$$H_{ij}^{(B)\dagger}(\vec{p}, \vec{x}) = H_{ij}^{(B)}(r(\vec{p}, \vec{x})), \quad (38)$$

$$C_{ij}^{(F)\dagger}(\vec{p}, \vec{x}) = C_{ij}^{(F)}(r(\vec{p}, \vec{x})) \quad \text{and} \quad (39)$$

$$C_{ij}^{(B)\dagger}(\vec{p}, \vec{x}) = C_{ij}^{(B)}(r(\vec{p}, \vec{x})). \quad (40)$$

Under this unitary/canonical transformation we, of course, also transform the wave function for the single particle or, equivalently, the fields into new fields. Thus we can now decide, for example, to use the physical momentum representation in which the “physical momentum” \vec{p} , defined below in subsection 4.2, is diagonal. The fields in this representation are naturally called $\phi_i(\vec{p})$ and $\psi_i(\vec{p})$. The symmetry relations (22-25) will also be valid when we use the “physical” momenta and positions instead of the $(\vec{p}_{pre}, \vec{x}_{pre})$ -variables.

In the following subsection 4.2 we shall discuss the way we are supposed to select the “physical” variables and thus which canonical transformation, or rather unitary transformation, should be performed, in order to go into the variables to be identified at the end with what we conceive to be the true momenta and positions.

4.2 The choice of what is physical position and momentum

We shall study below the spectrum of frequency eigenstates for the just discussed functions H and C , which formally look like functions defined on the phase space the coordinates of which are the pre-momenta and pre-positions. Quite abstractly and generally however, we may consider a spectrum of frequency eigenvalues ω to vary continuously over the phase space. Now the “poor physicist” has only very small energies at his disposal compared to the fundamental (presumably Planck) energy unit. So he only has access to the part of the spectrum which is very close to the Fermi-surface for fermions and close to zero frequency for the bosons. As one goes along in the phase space, the spectrum typically varies on a scale of the fundamental frequency. Therefore, in most of phase space, it will be tremendously faraway from the Fermi-surface or from zero from the point of view of the “poor physicist”. (We shall below argue that some boson frequencies remain exactly zero, but let us here for pedagogical reasons ignore these eigenvalues in the spectrum, which are anyway assumed not to be truly observed.) The eigenfrequencies which reach the Fermi-surface or the zero frequency surface do so on a sub-manifold of the phase space – which remarkably enough we shall show below to be of co-dimension 3 - and grow up along this sub-manifold linearly with momentum (the dominant Taylor expansion terms). Only a narrow band in phase space around this co-dimension 3 surface is relevant for the poor physicist.

Now the physical position and momentum, which henceforth will be denoted by \vec{x} and \vec{p} , are to be identified with linear combinations of pre-position and pre-momentum, such that the physical position varies along the bands whereas the physical momentum varies across the bands. To be more precise, we shall see below that in 3 directions the single particle frequencies increase linearly with the distance from a co-dimension 3 sub-manifold. Along this co-dimension 3 sub-manifold we have an extra $\omega = 0$ eigenvalue in the boson case and two eigenstates at just the Fermi-surface in the fermion case. It is these three directions along which the relevant ω ’s increase linearly that are to be identified with the physical momenta. We shall set the zero for the physical momentum to be just at the above-mentioned co-dimension 3 surface. The physical position must then be

defined to be along this co-dimension 3 surface. In general this co-dimension 3 surface will have more than 3 dimensions and, thus, the choice of the physical position space *a priori* now becomes arbitrary. However we can choose the genuine position directions to be determined from the physical momenta, by using them in a Poisson bracket to generate translations.

The physics that is relevant for the “poor physicist” is only the behaviour of the accessible part of the spectrum in the very narrow bands mentioned. We make the very general assumption that, in these bands, all the relevant functions H and C are Taylor expandable with high order terms being discarded.

Really the general procedure in our attempts to derive Lorentz invariance etc. is to use Taylor expansions, especially for the four functions we have written about here. For example we may use the Taylor expansion

$$H_{ij}^{(F)}(\vec{p}, \vec{x}) \approx H_{ij}^{(F)}(\vec{0}, \vec{x}) + \vec{p} \cdot \frac{\partial}{\partial \vec{p}} H_{ij}^{(F)}(\vec{p}, \vec{x})|_{\vec{p}=\vec{0}} + \dots \quad (41)$$

because the physical \vec{p} , with which the “poor physicist” can in practice work, is a very small quantity compared to the fundamental scale.

When the four operators we study here have to obey a reality condition like (26-29), the number of independent, say, first derivatives $\frac{\partial}{\partial \vec{p}} H_{ij}^{(F)}(\vec{p}, \vec{x})|_{\vec{p}=\vec{0}}$ gets lowered. Really the number of restrictions and thereby also the number of independent matrices in the expansion should depend in a smooth continuous way on the unitary transformation. Therefore, as one of course always expects an integer number of restrictions, this number must be independent of this deformation. In fact we may differentiate the equations (22-25) with respect to physical variables, since we remember that they are true also with the physical position and momentum, and we may differentiate the Hermiticity conditions (37-40) to get equations like

$$\frac{\partial}{\partial \vec{p}} C_{ij}^{(F)} = \frac{\partial}{\partial \vec{p}} C_{ji}^{(F)}, \quad (42)$$

$$\frac{\partial}{\partial \vec{p}} C_{ij}^{(F)\dagger}(\vec{p}, \vec{x}) = \frac{\partial}{\partial \vec{p}} C_{ij}^{(F)}(r(\vec{p}, \vec{x})). \quad (43)$$

There are quite analogous equations for the other three functions we consider.

5 Normalization of the fields ψ_i and ϕ_i by fixing the symmetric matrices to Kronecker deltas, equations of motion and spectra

It should be borne in mind that *a priori* the fields are arbitrarily normalized and that we may use the Hamiltonians to define the normalization of the fields, if we so choose. In fact an important ingredient in the formulation of the present work is to assume that a linear transformation has been made on the various field components $\phi_i(\vec{p})$, i.e. a transformation on the component index i , such that the symmetric coefficient functions $H_{ij}^{(B)}(\vec{p}, \vec{x})$ become equal to the unit matrix:

$$H_{ij}^{(B)}(\vec{p}, \vec{x}) = \delta_{ij} \quad (44)$$

for the very small momenta accessible to the poor physicist. Thereby, of course, the commutation relations among these components $\phi_i(\vec{p})$ are modified and we cannot simultaneously arrange for them to be trivial. So for the bosons we choose a notation in which the non-trivial behaviour of the equations of motion, as a function of the momentum \vec{p} , is put into the commutator expression

$$[\phi_i(\vec{p}), \phi_j(\vec{p}')] = iC_{ij}^{(B)}(\vec{p}, \vec{x})\delta(\vec{p} - \vec{p}') \quad (45)$$

It follows that the information which we would, at first, imagine should be contained in the Hamiltonian is, in fact, now contained in the antisymmetric matrices $C_{ij}^{(B)}(\vec{p}, \vec{x})$.

This means that we normalize the fields so as to make the Hamiltonian become

$$H_B = \frac{1}{2} \int \sum_i \phi_i(\vec{p}) \phi_i(\vec{p}) d\vec{p}. \quad (46)$$

In principle we could also have used the same normalization procedure for the fermions. However, in both cases, we rather choose the symmetric functions among our four C_{ij} and H_{ij} functions to be normalized to the Kronecker deltas in Eq. (44). So, for the fermions, we just take $C_{ij}^{(F)} = \delta_{ij}$. This means that for the fermions, we shall keep to the more usual formulation. In other words, we normalize the anti-commutator to be the unit matrix and let the more non-trivial dependence on \vec{p} sit in the Hamiltonian coefficient functions $H_{ij}^{(F)}(\vec{p}, \vec{x})$. That is to say that we have the usual equal time anti-commutation relations:

$$\{\psi_i(\vec{p}), \psi_j(\vec{p}')\} = \delta_{ij} \delta(\vec{p} - \vec{p}'). \quad (47)$$

The component indices i, j enumerate the very general discrete degrees of freedom in the model. These degrees of freedom might, at the end, be identified with Hermitian and anti-Hermitian components, spin components, variables versus conjugate momenta or even totally different types of particle species, such as flavours etc. It is important to realize that this model is so general that it has, in that sense, almost no assumptions built into it—except for our free approximation, the above-mentioned rudimentary momentum conservation and some general features of second quantized models. It follows from the rudimentary momentum conservation in our model that the (anti-)commutation relations have a $\delta(\vec{p} - \vec{p}')$ delta function factor in them.

5.1 Equations of motion for the general fields

We can easily write down the equations of motion for the field components in our general quantum field theory, both in the fermionic case:

$$\dot{\psi}_i(\vec{p}) = i[H_F, \psi_i(\vec{p})] = - \sum_k \psi_k(\vec{p}) H_{ki}^{(F)}(\vec{p}, \vec{x}) \quad (48)$$

and in the bosonic case:

$$\dot{\phi}_i(\vec{p}) = i[H_B, \phi_i(\vec{p})] = - \sum_k \phi_k(\vec{p}) C_{ki}^{(B)}(\vec{p}, \vec{x}). \quad (49)$$

We see that both the bosonic and the fermionic equations of motion are of the form

$$\dot{\xi}_i(\vec{p}) = \sum_k A_{ik} \xi_k(\vec{p}) \quad (50)$$

Here $A_{ij}(\vec{p}, \vec{x})$ is a real antisymmetric matrix and we have introduced the neutral name ξ for both the boson field ϕ and the fermion field ψ .

5.2 Spectrum of an antisymmetric matrix $A_{ij}(\vec{p}, \vec{x})$

An antisymmetric matrix $A_{ij}(\vec{p}, \vec{x})$ with real matrix elements is anti-Hermitian and thus has purely imaginary eigenvalues. However, if we look for a time dependence ansatz of the form

$$\xi_i(\vec{p}, t) = a_i(\vec{p}) \exp(-i\omega t), \quad (51)$$

the eigenvalue equation for the frequency ω becomes

$$\omega a_i = \sum_j iA_{ij}(\vec{p}, \vec{x}) a_j. \quad (52)$$

Now the matrix $iA_{ij}(\vec{p}, \vec{x})$ is Hermitian and the eigenvalues ω are therefore real.

It is easy to see, that if ω is an eigenvalue, then so also is $-\omega$. In fact we could imagine calculating the eigenvalues by solving the equation

$$\det(iA - \omega) = 0. \quad (53)$$

We then remark that transposition of the matrix $(iA - \omega)$ under the determinant sign will not change the value of the determinant, but corresponds to changing the sign of ω because of the antisymmetry of the matrix iA . So non-vanishing eigenvalues occur in pairs.

Now the main point of interest for our study is how the second quantized model looks close to its ground state. The neighbourhood of this ground state is supposed to be the only regime which we humans can study in our “low energy” experiments, with small momenta compared to the fundamental (say Planck) mass scale. With respect to the ground state of such a second quantized world machinery, it is well-known that there is a difference between the fermionic and the bosonic case. In the fermionic case, you can at most have one fermion in each state and must fill the states up to some specific value of the single particle energy—which is really ω . However, in the bosonic case, one can put a large number of bosons into the same orbit/single particle state, if that should pay energetically.

5.3 The vacuum

If we allow for the existence of a chemical potential, which essentially corresponds to the conservation of the number of fermions, we shall typically get the ground state to have fermions filling the single particle states up to some special value of the energy called the Fermi-energy ω_{FS} (FS standing for “Fermi-surface”). For bosons, on the other hand, we will always have zero particles in all the orbits, except perhaps in the zero energy ground state; it will namely never pay energetically to put any bosons into positive energy orbits.

5.4 The lowest excitations

So for the investigation of the lowest excitations, i.e. those that a “poor physicist” could afford to work with, we should look for the excitations very near to the Fermi-surface in the fermionic case. In other words, we should put fermions into the orbits with energies very little above the Fermi-energy, or make holes in the Fermi-sea at values of the orbit-energies very little below the Fermi-energy. Thus, for excitations accessible to the “poor physicist”, it is only necessary to study the behaviour of the spectrum for the bosons having a value of ω near to zero, and for the fermions having a value of ω near the Fermi-energy ω_{FS} .

6 Jahn-Teller effect and analogue for bosons

The Jahn-Teller effect [3] refers to a very general feature of systems of fermions, which possess some degrees of freedom that can adjust themselves so as to lower the energy as much as possible. The effect is so general that it should be useful for almost all systems of fermions, because even if they did not have any extra degrees of freedom to adjust there would, in the Hartree approximation, be the possibility that the fermions could effectively adjust themselves. The name Homolumo gap was introduced for this effect in chemistry and stands for the gap between “the highest occupied”

HO “molecular orbit” MO and the “lowest unoccupied” LU “molecular orbit” MO. The point is simply that if the filled (occupied) orbits (single particle states) are lowered the whole energy is lowered, while it does not help to lower the empty orbits. It therefore pays energetically to make the occupied orbits go down in energy and separate from the unfilled ones; thus a gap may appear or rather *there will be a general tendency to get a low level density near the Fermi-surface*. This effect can easily be so strong that it causes a symmetry to break [3]; symmetry breaking occurs if some levels, which are degenerate due to the symmetry, are only partially filled so that the Fermi-surface just cuts a set of degenerate states/orbits. It is also the Homolumo-gap effect which causes the deformation of transitional nuclei, which are far from closed shell configurations. We want to appeal to this Homolumo gap effect, in section 8, as a justification for the assumption that the Fermi-surface gets close to those places on the energy axis where the level density is minimal.

However we first want to discuss a similar effect, where the degrees of freedom of a system of bosons adjust themselves to lower the total energy. As for the fermion systems just discussed, this lowering of the total energy is due to the adjustment of a sum over single particle energies—the minimisation of the zero-point energy of the bosonic system. We consider the effect of this minimisation to be the analogue for bosons of the Jahn-Teller effect.

6.1 The analogue for bosons

Let us imagine that there are many degrees of freedom of the whole world machinery that could adjust themselves (we called them the “rest” degrees of freedom in section 2) to minimize the energy of the system and could also influence the matrix $A_{ij}(\vec{p}, \vec{x})$. Then one could, for instance, ask how it would be energetically profitable to adjust the eigenvalues, in order to minimize the zero-point energy of the whole (second quantized) system. This zero-point energy is formally given by the integral over all (the more than three dimensional) momentum space; let us just denote this integration measure by $d\vec{p}$, so that:

$$E_{\text{zero-point}} = \int d\vec{p} \sum_{\text{eigenvalue pair } k} |\omega_k(\vec{p})|/2. \quad (54)$$

Provided some adjustment took place in minimizing this quantity, there would *a priori* be an argument in favour of having several zero eigenvalues, since they would contribute the least to this zero-point energy $E_{\text{zero-point}}$. At first sight, this argument is not very strong, since it just favours making the eigenvalues small and not necessarily making any one of them exactly zero. However, we underlined an important point in favour of the occurrence of exactly zero eigenvalues, by putting the numerical sign explicitly into the integrand $|\omega_k(\vec{p})|/2$ in the expression (54) for the zero-point energy. The important point is that the numerical value function is not an ordinary analytic function, but rather has a kink at $\omega_k(\vec{p}) = 0$. This means that, if other contributions to the energy of the whole system are smooth/analytic, it could happen that the energy is lowered when $\omega_k(\vec{p})$ is lowered numerically for both signs of $\omega_k(\vec{p})$; here we consider $\omega_k(\vec{p})$ to be a smooth function of the adjusting parameters of the whole world machinery (the “rest” degrees of freedom). For a normal analytic energy function this phenomenon could of course never occur, except if the derivative just happened (is fine-tuned one could say) to be equal to zero at $\omega_k(\vec{p}) = 0$. But with a contribution that has the numerical value singularity behaviour it is possible to occur with a finite probability (i.e. without fine-tuning), because it is sufficient that the derivative of the contribution to the total energy from other terms is numerically lower than the derivative of the zero-point term discussed. Then, namely, the latter will dominate the sign of the slope and the minimum will occur *exactly* for zero $\omega_k(\vec{p})$.

In this way, we claim to justify our assumption in the following section that the matrix $A_{ij}(\vec{p}, \vec{x})$ will have several, say n exactly zero eigenvalues and thus a far from maximal rank; the rank being

at least piecewise constant over momentum space. We shall therefore now study antisymmetric matrices with this property in general and look for their lowest energy excitations.

7 Derivation of Lorentz invariance and the free Maxwell equations

It can happen that, for special values of the “momentum parameters”, a pair of eigenvalues of the antisymmetric matrix $A_{ij}(\vec{p}, \vec{x})$ —consisting of eigenvalues of opposite sign of course—approach the group of exactly zero eigenvalues. It is this situation which we believe to be the one of relevance for the low energy excitations. We shall therefore concentrate our interest on a small region in the momentum parameter space, around a point \vec{p}_0 where the two levels with the numerically smallest non-zero eigenvalues merge together with n levels having zero eigenvalue. Following the notation for physical momentum introduced in section (4.2), we set $\vec{p}_0 = 0$. Using the well-known fact that, in quantum mechanics, perturbation corrections from faraway levels have very little influence on the perturbation of a certain level, we can ignore all the levels except the n zero eigenvalues and this lowest non-zero pair.

Hence we end up with an effective $(n+2) \times (n+2)$ matrix $A_{ij}(\vec{p}, \vec{x})$ obeying the constraint of being of rank two (at most). Now we imagine that we linearize the momentum dependence of $A_{ij}(\vec{p}, \vec{x})$ on \vec{p} around a point in momentum space, $\vec{p}_0 = 0$, where the pair of eigenvalues approaching zero actually reach zero, so that the matrix is totally zero, $A_{ij}(\vec{p}_0, \vec{x}) = 0$, at the starting point for the Taylor expansion. That is to say that, corresponding to different basis vectors in momentum space, we get contributions to the matrix $A_{ij}(\vec{p}, \vec{x})$ linear in the momentum difference $\vec{p} - \vec{p}_0 = \vec{p}$. Now any non-zero antisymmetric matrix is necessarily of rank at least 2. So the contribution from the first chosen basis vector in momentum space will already give a matrix A_{ij} of rank 2 and contributions from other momentum components should not increase the rank beyond this. A single basis vector for a set of linearly parameterised antisymmetric real matrices can be transformed to just having elements (1,2) and (2,1) nonzero and the rest zero. In order to avoid a further increase in the rank of the matrix by adding other linear contributions, these further contributions must clearly not contribute anything to matrix elements having both column and row index different from 1 and 2. However this is not sufficient to guarantee that the rank remains equal to 2. This is easily seen, because we can construct 4×4 antisymmetric matrices, which are of the form of having 0's on all places (i,j) with both i and j different from 1 and 2 and have non-zero determinant.

So let us consider 4 by 4 sub-determinants of the matrix A_{ij} already argued to be of the form

$$\begin{pmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1n} \\ -A_{12} & 0 & A_{23} & \cdots & A_{2n} \\ -A_{13} & -A_{23} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & 0 \\ -A_{1n} & -A_{2n} & 0 & \cdots & 0 \end{pmatrix}. \quad (55)$$

Especially let us consider a four by four sub-determinant along the diagonal involving columns and rows 1 and 2. The determinant is for instance

$$\det \begin{pmatrix} 0 & A_{12} & A_{13} & A_{15} \\ -A_{12} & 0 & A_{23} & A_{25} \\ -A_{13} & -A_{23} & 0 & 0 \\ -A_{15} & -A_{25} & 0 & 0 \end{pmatrix} = - \left(\det \begin{pmatrix} A_{13} & A_{15} \\ A_{23} & A_{25} \end{pmatrix} \right)^2. \quad (56)$$

In order that the matrix A_{ij} be of rank 2, this determinant must vanish and so we require that

the 2 by 2 sub-matrix

$$\left(\begin{array}{c|c} A_{13} & A_{15} \\ A_{23} & A_{25} \end{array} \right) \quad (57)$$

must be degenerate, i.e. of rank 1 only. This means that the two columns in it are proportional, one to the other. By considering successively several such selected four by four sub-matrices, we can easily deduce that all the two columns

$$\left(\begin{array}{c} A_{13} \\ A_{23} \end{array} \right), \left(\begin{array}{c} A_{14} \\ A_{24} \end{array} \right), \dots \left(\begin{array}{c} A_{1n} \\ A_{2n} \end{array} \right) \quad (58)$$

are proportional. This in turn means that we can transform them all to zero, except for say

$$\left(\begin{array}{c} A_{13} \\ A_{23} \end{array} \right), \quad (59)$$

by going into a new basis for the fields $\phi_k(\vec{p})$. So, finally, we have transformed the formulation of the fields in such a way that only the upper left three by three corner of the A matrix is non-zero.

This, in turn, means that we can treat the bosonic model in the region of interest, by studying the spectrum of a (generic) antisymmetric 3×3 matrix with real elements, or rather such a matrix multiplied by i . Let us immediately notice that such a matrix is parameterised by three parameters. The matrix and thus the spectrum, to the accuracy we are after, can only depend on three of the momentum parameters. In other words the dispersion relation will depend trivially on all but 3 parameters in the linear approximation. By this linear approximation, we here mean the approximation in which the “poor physicist” can only work with a small region in momentum parameter space also—not only in energy. In this region we can trust the lowest order Taylor expansion in the differences of the momentum parameters from their starting values (where the nearest levels merge). Then the ω -eigenvalues—i.e. the dispersion relation—will not vary in the direction of a certain subspace of co-dimension three. Corresponding to these directions the velocity components of the described boson particle will therefore be zero! The boson, as seen by the “poor physicist”, can only move inside a three dimensional space; in other directions its velocity must remain zero. It is in this sense we say that the three-dimensionality of space is explained!

The form of the equations of motion for the bosonic fields, in this low excitation regime where one can use the lowest order Taylor expansion in the momentum parameters, is also quite remarkable: after a linear transformation in the space of “momentum parameters”, they can be transformed into the *Maxwell equations* with the fields being complex (linear) combinations of the magnetic and electric fields. Let us now derive this result.

The lowest order Taylor expansion of the antisymmetric matrix $A_{ij}(\vec{p}, \vec{x})$ in \vec{p} , with \vec{x} -dependent coefficients, takes the form:

$$A = \left(\begin{array}{c|c|c} 0 & c^{3i}(\vec{x})p_i & -c^{2i}(\vec{x})p_i \\ -c^{3i}(\vec{x})p_i & 0 & c^{1i}(\vec{x})p_i \\ c^{2i}(\vec{x})p_i & -c^{1i}(\vec{x})p_i & 0 \end{array} \right). \quad (60)$$

Its eigenvalues are $-i\omega = 0, \pm i\sqrt{g^{ij}(\vec{x})p_i p_j}$, where the real symmetric 3×3 spatial metric tensor is:

$$g_{Maxwell}^{ij}(\vec{x}) = \sum_{t=1}^3 c^{ti}(\vec{x})c^{tj}(\vec{x}). \quad (61)$$

Since, at the outset, our model is pre-Einsteinian in the sense of having an absolute time, we can form a 4×4 space-time metric $g_{Maxwell}^{\mu\nu}$, by supplementing equation (61) with the definitions:

$$g_{Maxwell}^{00} = 1 \quad \text{and} \quad g_{Maxwell}^{0i} = g_{Maxwell}^{i0} = 0. \quad (62)$$

It is possible to construct the matrix square root $\sqrt{g_{\text{Maxwell}}^{ij}}$ of the spatial metric tensor (61) and, by making an orthogonal transformation on the fields $\phi_k(\vec{p})$, we can choose a basis in which $A_{ij}(\vec{p}, \vec{x})$ becomes:

$$A = \left(\begin{array}{c|c|c} 0 & \sqrt{g_{\text{Maxwell}}^{3i}}(\vec{x})p_i & -\sqrt{g_{\text{Maxwell}}^{2i}}(\vec{x})p_i \\ -\sqrt{g_{\text{Maxwell}}^{3i}}(\vec{x})p_i & 0 & \sqrt{g_{\text{Maxwell}}^{1i}}(\vec{x})p_i \\ \sqrt{g_{\text{Maxwell}}^{2i}}(\vec{x})p_i & -\sqrt{g_{\text{Maxwell}}^{1i}}(\vec{x})p_i & 0 \end{array} \right). \quad (63)$$

A priori the spatial metric is \vec{x} -dependent. However we speculate that, if gravity can be incorporated into our theory, the spatial metric becomes flat for some dynamical reason (cosmological inflation?). Henceforth we assume that translational invariance arises in this way and thus obtain:

$$A = \left(\begin{array}{c|c|c} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{array} \right) \quad (64)$$

in the chosen basis.

We can now make a Fourier transform of the three fields $\phi_j(\vec{p})$ into the \vec{x} -representation. These new position space fields $\phi_j(\vec{x})$ are no longer Hermitian. However, it follows from the assumed Hermiticity of the $\phi_j(\vec{p})$ that, in the \vec{x} -representation, the real parts of the fields $\phi_j(\vec{x})$ are even, while the imaginary parts are odd functions of \vec{x} :

$$\vec{\phi}(\vec{x}) = \vec{\phi}^\dagger(-\vec{x}). \quad (65)$$

We now want to extract the real and imaginary parts and identify them with the magnetic \vec{B} and electric \vec{E} fields. This can be done in many ways, in as far as we could take

$$e^{i\delta}\vec{\phi}(\vec{x}) = \vec{B}(\vec{x}) + i\vec{E}(\vec{x}) \quad (66)$$

with an arbitrary phase δ . For definiteness we shall, however, choose an especially nice identification with the property that the field $\vec{\phi}(\vec{x})$ constructed from \vec{E} and \vec{B} has simple transformation properties under P , T and PT :

$$P\vec{\phi}(\vec{x})P^{-1} = \vec{\phi}^\dagger(-\vec{x}), \quad T\vec{\phi}(\vec{x})T^{-1} = \vec{\phi}(\vec{x}) \quad \text{and} \quad PT\vec{\phi}(\vec{x})(PT)^{-1} = \vec{\phi}^\dagger(-\vec{x}). \quad (67)$$

It then follows from (65) that

$$PT\vec{\phi}(\vec{x})(PT)^{-1} = \vec{\phi}(\vec{x}) \quad \text{and} \quad P\vec{\phi}(\vec{x})P^{-1} = \vec{\phi}(\vec{x}). \quad (68)$$

These two expressions (68) are, of course, equivalent to each other, due to the relation $T\vec{\phi}(\vec{x})T^{-1} = \vec{\phi}(\vec{x})$. Since under time reversal

$$\vec{E}(\vec{x}) \rightarrow -\vec{E}(\vec{x}), \quad \vec{B}(\vec{x}) \rightarrow \vec{B}(\vec{x}), \quad i \rightarrow -i, \quad (69)$$

it is clear that to satisfy (68), we should make the identification

$$\vec{\phi}(\vec{x}) = \vec{B}(\vec{x}) + i\vec{E}(\vec{x}) \quad (70)$$

as in [2].

Thus we identify the real and imaginary parts of $\vec{\phi}(\vec{x})$ as magnetic and electric fields $\vec{B}(\vec{x})$ and $\vec{E}(\vec{x})$ respectively: $\phi_j(\vec{x}) = iE_j(\vec{x}) + B_j(\vec{x})$. However the symmetry of these Maxwell fields means that they must be in a configuration/state which goes into itself under a parity reflection in the origin. This is a somewhat strange feature which seems necessary for the identification of our

general fields with the Maxwell fields; we discuss this feature further in section 12. For the moment let us, however, see that we do indeed get the Maxwell equations in the free approximation with the proposed identification.

By making the inverse Fourier transformation back to momentum space, we obtain the following identification of the fields $\phi_j(\vec{p})$ in our general quantum field theory with the electric field $E_j(\vec{p})$ and magnetic field $B_j(\vec{p})$ Fourier transformed into momentum space:

$$\begin{pmatrix} \phi_1(\vec{p}) \\ \phi_2(\vec{p}) \\ \phi_3(\vec{p}) \end{pmatrix} = \begin{pmatrix} iE_1(\vec{p}) + B_1(\vec{p}) \\ iE_2(\vec{p}) + B_2(\vec{p}) \\ iE_3(\vec{p}) + B_3(\vec{p}) \end{pmatrix}. \quad (71)$$

We note that the Fourier transformed electric field $E_j(\vec{p})$ in the above ansatz (71) has to be purely imaginary, while the magnetic field $B_j(\vec{p})$ must be purely real.

By using the above identifications, eqs. (64) and (71), the equations of motion (49) take the following form

$$\begin{pmatrix} i\dot{E}_1(\vec{p}) + \dot{B}_1(\vec{p}) \\ i\dot{E}_2(\vec{p}) + \dot{B}_2(\vec{p}) \\ i\dot{E}_3(\vec{p}) + \dot{B}_3(\vec{p}) \end{pmatrix} = \begin{pmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{pmatrix} \begin{pmatrix} iE_1(\vec{p}) + B_1(\vec{p}) \\ iE_2(\vec{p}) + B_2(\vec{p}) \\ iE_3(\vec{p}) + B_3(\vec{p}) \end{pmatrix}. \quad (72)$$

We can now use the usual Fourier transformation identification in quantum mechanics to transform these equations to the \vec{x} -representation, simply from the definition of \vec{x} as the Fourier transformed variable set associated with \vec{p} ,

$$p_j = i^{-1}\partial_j. \quad (73)$$

Thus in \vec{x} -representation the equations of motion become

$$\begin{pmatrix} i\dot{E}_1(\vec{x}) + \dot{B}_1(\vec{x}) \\ i\dot{E}_2(\vec{x}) + \dot{B}_2(\vec{x}) \\ i\dot{E}_3(\vec{x}) + \dot{B}_3(\vec{x}) \end{pmatrix} = \begin{pmatrix} 0 & -i\partial_3 & i\partial_2 \\ i\partial_3 & 0 & -i\partial_1 \\ -i\partial_2 & i\partial_1 & 0 \end{pmatrix} \begin{pmatrix} iE_1(\vec{x}) + B_1(\vec{x}) \\ iE_2(\vec{x}) + B_2(\vec{x}) \\ iE_3(\vec{x}) + B_3(\vec{x}) \end{pmatrix}. \quad (74)$$

The imaginary terms in the above equations give rise to the equation

$$\dot{\vec{E}}(\vec{x}) = \text{curl } \vec{B}(\vec{x}), \quad (75)$$

while the real parts give the equation:

$$\dot{\vec{B}}(\vec{x}) = -\text{curl } \vec{E}(\vec{x}). \quad (76)$$

These two equations are just the Maxwell equations in the absence of charges and currents, except that strictly speaking we miss two of the Maxwell equations, namely

$$\text{div } \vec{E}(\vec{x}) = 0 \quad \text{and} \quad \text{div } \vec{B}(\vec{x}) = 0. \quad (77)$$

However, these two missing equations are derivable from the other Maxwell equations in time differentiated form. That is to say, by using the result that the divergence of a curl is zero, one can derive from the other equations that

$$\text{div } \dot{\vec{E}}(\vec{x}) = 0 \quad \text{and} \quad \text{div } \dot{\vec{B}}(\vec{x}) = 0 \quad (78)$$

which is though not quite sufficient. Integration of the resulting equations (78) effectively replaces the 0's on the right hand sides of equations (77) by terms constant in time, which we might interpret as some constant electric and magnetic charge distributions respectively. In our free

field theory approximation, we have potentially ignored such terms. So we may claim that, in the approximation to which we have worked so far, we have derived the Maxwell equations sufficiently well. We shall consider interactions with the fermion field in section 10.

For later use, it is now convenient to introduce the usual relativistic notation for the second rank antisymmetric electromagnetic field $F^{\alpha\beta}$ tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\text{Im } \phi_1 & -\text{Im } \phi_2 & -\text{Im } \phi_3 \\ \text{Im } \phi_1 & 0 & -\text{Re } \phi_3 & \text{Re } \phi_2 \\ \text{Im } \phi_2 & \text{Re } \phi_3 & 0 & -\text{Re } \phi_1 \\ \text{Im } \phi_3 & -\text{Re } \phi_2 & \text{Re } \phi_1 & 0 \end{pmatrix} \quad (79)$$

and the corresponding dual tensor

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad \text{where} \quad F_{\gamma\delta} = g_{\alpha\gamma} F^{\alpha\beta} g_{\delta\beta}. \quad (80)$$

Here $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric symbol in four indices and $g_{\mu\nu} = g_{\mu\nu}^{Maxwell}$, as defined in equations (61) and (62) (still assuming flatness), is the usual metric of special relativity. Thus we have:

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -\text{Re } \phi_1 & -\text{Re } \phi_2 & -\text{Re } \phi_3 \\ \text{Re } \phi_1 & 0 & \text{Im } \phi_3 & -\text{Im } \phi_2 \\ \text{Re } \phi_2 & -\text{Im } \phi_3 & 0 & \text{Im } \phi_1 \\ \text{Re } \phi_3 & \text{Im } \phi_2 & -\text{Im } \phi_1 & 0 \end{pmatrix} \quad (81)$$

By constructing the “self-dual” combination we get

$$F^{\alpha\beta} + i\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -i\phi_1^\dagger & -i\phi_2^\dagger & -i\phi_3^\dagger \\ i\phi_1^\dagger & 0 & -\phi_3^\dagger & \phi_2^\dagger \\ i\phi_2^\dagger & \phi_3^\dagger & 0 & -\phi_1^\dagger \\ i\phi_3^\dagger & -\phi_2^\dagger & \phi_1^\dagger & 0 \end{pmatrix} \quad (82)$$

and clearly the conjugate “anti self-dual” combination is then

$$F^{\alpha\beta} - i\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & i\phi_1 & i\phi_2 & i\phi_3 \\ -i\phi_1 & 0 & -\phi_3 & \phi_2 \\ -i\phi_2 & \phi_3 & 0 & -\phi_1 \\ -i\phi_3 & -\phi_2 & \phi_1 & 0 \end{pmatrix}. \quad (83)$$

8 Derivation of the Weyl equation

Let us now turn to the application of the Jahn-Teller or Homolumo-gap effect to a system of fermions in our general field theory model. We shall assume that the Homolumo-gap effect turns out to be strong enough to ensure that the Fermi-surface just gets put to a place where the density of levels is very low. Actually it is very realistic that a gap should develop in a field theory with continuum variables \vec{p} labelling the single particle states. That is namely what one actually sees in an insulator; there is an appreciable gap between the last filled *band* and the first empty band. However, if the model were totally of this insulating type, the poor physicist would not “see” anything, because he is supposed to be unable to afford to raise a particle from the filled band to the empty one. So he can only see something if there are at least *some* fermion single particle states with energy close to the Fermi-surface.

We shall now divide up our discussion of what happens near the Fermi-surface according to the number of components of the fermion field that are relevant in this neighbourhood. Let us denote

by n the number of fermion field components, which contribute significantly to the eigenstates near the Fermi-surface in the small region of momentum space we choose to consider.

We shall see that, instead of working with the antisymmetric matrix with purely imaginary matrix elements iA_{ij} , near the Fermi surface we can work rather with asymmetric Hermitian matrices. In fact we have already noted that the eigenvalues for the matrix iA_{ij} all the time occur together as pairs $\pm\omega$. Thus, if at first we want to consider matrices of the antisymmetric form iA_{ij} with levels near the Fermi surface, we must also keep the corresponding ones having eigenvalues nearly equal to minus the Fermi energy. So if, after replacing the matrix by an effective one, we want to work near the Fermi surface, we should keep those eigenvalues in the matrix which are both near the Fermi surface and near the opposite of the Fermi energy. Thus, taking the number of relevant eigenvalues near the Fermi surface to be n , the number of rows and of columns to be kept in the antisymmetric iA_{ij} formalism should be $2n$. The matrix $iA_{ij}^{nearF.S.}$, with which we then replace iA_{ij} near the Fermi surface, is a $2n$ by $2n$ matrix that is very close to the matrix with the n -times degenerate eigenvalue ω_{FS} , i.e. the Fermi energy, and with the other n eigenvalues also collapsing to one n -times degenerate eigenvalue $-\omega_{FS}$. Thus, to the crudest approximation, this matrix $iA_{ij}^{nearF.S.}$ is proportional to the matrix \mathbf{J} consisting of four blocks, with the n by n unit matrix and minus the n by n unit matrix in the off-diagonal blocks and zero in the diagonal blocks. At a certain point \vec{p}_0 in momentum space (or in phase space if we think in terms of phase space), we shall in fact take the n eigenvalues to be just the Fermi energy ω_{FS} and the other n to be $-\omega_{FS}$. When we move a little away from this point, *a priori* terms linear in a Taylor expansion around that point will appear all over the $2n \times 2n$ matrix $i\{A_{ij}^{nearF.S.}\} = i\mathbf{A}^{nearF.S.}$.

$$i\mathbf{A}^{nearF.S.} \approx i\omega_{FS}\mathbf{J} + i\frac{\partial\mathbf{A}^{nearF.S.}}{\partial p_i}(p_i - p_{0i}) = i\omega_{FS} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} + i\frac{\partial\mathbf{A}^{nearF.S.}}{\partial p_i}(p_i - p_{0i}) \quad (84)$$

Parts of the linear term $i\frac{\partial\mathbf{A}^{nearF.S.}}{\partial p_i}(p_i - p_{0i})$ which, as matrices, do not commute with \mathbf{J} will only contribute at second order in perturbation theory. So we shall ignore their effects and only include terms contributing to $iA_{ij}^{nearF.S.}$ which have the property of commuting with \mathbf{J} . Such contributions are characterized by having the two n by n diagonal blocks being identical and the two off-diagonal blocks being equal except for an overall sign. Since the matrix shall be of the same type as $iA_{ij}^{nearF.S.}$, i.e. purely imaginary and antisymmetric, the n by n diagonal blocks are antisymmetric while the n by n off-diagonal blocks are symmetric. There are thus clearly n^2 free real variables. This is the same number as for an n by n Hermitian complex matrix. We can indeed easily make a correspondence between the type of linear contributions to $iA_{ij}^{nearF.S.}$ which we have to consider and $n \times n$ Hermitian matrices. This linear contribution takes the form

$$iA_{ij} = i \begin{pmatrix} A_S^{n \times n} & A_A^{n \times n} \\ -A_S^{n \times n} & A_A^{n \times n} \end{pmatrix} = A_S^{n \times n} \otimes (-\sigma^2) + A_A^{n \times n} \otimes i\sigma^0 \quad (85)$$

and the corresponding Hermitian matrix is given by

$$H = A_S^{n \times n} + iA_A^{n \times n}, \quad (86)$$

where $A_S^{n \times n}$ and $A_A^{n \times n}$ denote symmetric and antisymmetric matrices respectively. In this way we basically interpret the matrix \mathbf{J} as the imaginary element i . For the purpose of calculating the positive eigenvalues ω near ω_{FS} , we can replace the purely imaginary antisymmetric matrix $i\mathbf{A}^{nearF.S.}$ by the Hermitian matrix H of equation (86).

So now let us consider the level-density near the Fermi-surface for n complex fermion field components.

8.1 The case of $n = 0$ relevant levels near the Fermi-surface

The $n = 0$ case must, of course, mean that there are no levels at all near the Fermi-surface in the small momentum range considered. This corresponds to the already mentioned insulator case. The poor physicist sees nothing from such regions in momentum space and he will not care for such regions at all. Nonetheless this is the generic situation close to the Fermi surface and will apply for most of the momentum space.

8.2 The case of $n = 1$ single relevant level near the Fermi-surface

In this case the generic situation will be that, as a certain component of the momentum is varied, the level will vary continuously in energy. This is the kind of behaviour observed in a metal. So there will be a rather smooth density of levels and such a situation is not favoured by the Homolumo gap effect, if there is any way to avoid it.

8.3 The case of $n = 2$ relevant levels near the Fermi-surface

In this situation a small but almost trivial calculation is needed. We must estimate how a Hamiltonian, described effectively as a 2 by 2 Hermitian matrix H with matrix elements depending on the momentum \vec{p} , comes to look in the generic case—i.e. when nothing is fine-tuned—and especially how the level density behaves. That is, however, quite easily seen, when one remembers that the three Pauli matrices and the unit 2 by 2 matrix together form a basis for the four dimensional space of two by two matrices. All possible Hermitian 2 by 2 matrices can be expressed as linear combinations of the three Pauli matrices σ^j and the unit 2 by 2 matrix σ^0 with real coefficients. We now consider a linearized Taylor expansion³ of the momentum dependence of such matrices, by taking the four coefficients to these four matrices to be arbitrary linear functions of the momentum minus the “starting momentum” \vec{p}_0 , where the two levels become degenerate with energy $\omega(\vec{p}_0)$. That is to say we must take the Hermitian 2 by 2 matrix to be

$$H = \sigma^a V_a^i(p_i - p_{0i}) + \sigma^0 \omega(\vec{p}_0) = \sigma^a V_a^i(\vec{x})(p_i - A_i(\vec{x})) + \sigma^0 A_0(\vec{x}) \quad (87)$$

where we formally identify the \vec{p}_0 contributions as a four-potential $A_a(\vec{x}) = V_a^i(\vec{x})A_i(\vec{x})$. This can actually be interpreted as the Hamiltonian for a particle obeying the Weyl equation, by defining the “covariant” momenta

$$P_1 = V_1^i(\vec{x})(p_i - A_i(\vec{x})), \quad P_2 = V_2^i(\vec{x})(p_i - A_i(\vec{x})), \quad P_3 = V_3^i(\vec{x})(p_i - A_i(\vec{x})), \quad (88)$$

and the renormalised Hamiltonian

$$H_{new} = H - \sigma^0 V_0^i(\vec{x})(p_i - A_i(\vec{x})) - \sigma^0 A_0(\vec{x}) = \vec{\sigma} \cdot \vec{P}, \quad (89)$$

$$\omega_{new} = \omega - V_0^i(\vec{x})(p_i - A_i(\vec{x})) - A_0(\vec{x}). \quad (90)$$

For simplicity, we shall here assume translational invariance and take the $V_a^i(\vec{x})$, $A_i(\vec{x})$ and $A_0(\vec{x})$ to be constants independent of position. The renormalisation of the energy, eq. (90), is then the result of transforming away a constant velocity V_0^i in d dimensions carried by all the fermions, using the change of co-ordinates $x'^i = x^i - tV_0^i$, and measuring the energy relative to $\omega(\vec{p}_0)$. Note that the “starting momentum” \vec{p}_0 will generically be of the order of a fundamental (Planck scale) momentum, which cannot be significantly changed by a “poor physicist”. So the large momentum \vec{p}_0 effectively plays the role of a conserved charge at low energy.

³A related discussion of the redefinition of spinors has been given in the context of the low energy limit of a Lorentz violating QED model [4].

A trivial calculation for the Weyl equation, $H_{new}\psi = \omega_{new}\psi$, leads to a level density with a thin neck, behaving like

$$\rho \propto \omega_{new}^2. \quad (91)$$

According to our strong assumption about Homolumo-gap effects, we should therefore imagine that the Fermi-surface in this case would adjust itself to be near the $\omega_{new} = 0$ level. Thereby there would then be the fewest levels near the Fermi-surface.

8.4 The cases $n \geq 3$

For n larger than 2 one can easily find out that, in the neighbourhood of a point where the n by n general Hamiltonian matrix deviates by zero from the unit matrix, there are generically branches of the dispersion relation for the particle states that behave in the metallic way locally, as in the case $n = 1$. This means that the level density in such a neighborhood has contributions like that in the $n = 1$ case, varying rather smoothly and flatly as a function of ω . So these cases are not so favourable from the Homolumo-gap point of view.

8.5 Conclusion of the various n cases for the fermion model

The conclusion of the just given discussion of the various n -cases is that, while of course the $n = 0$ case is the “best” case from the point of view of the Jahn-Teller or Homolumo-gap effect, it would not be noticed by the “poor physicist” and thus would not be of any relevance for him. The next “best” from the Homolumo-gap point of view is the case $n = 2$ of just two complex components (corresponding to 4 real components) being relevant near the Fermi-surface. Then there is a neck in the distribution of the levels, which is not present in the cases $n = 1$ and $n > 2$.

So the “poor physicist” should in practice observe the case $n = 2$, provided the Homolumo-gap effect is sufficiently strong (a perhaps suspicious assumption).

Now, as we saw, this case of $n = 2$ means that the fermion field satisfies a Weyl equation, formally looking like the Weyl equation in just 3+1 dimensions! It should however be noticed that there are indeed more spatial dimensions, by assumption, in our model. In these extra spatial dimensions, the fermions have the same constant velocity which we were able to renormalise to zero, because the Hamiltonian only depends on the three momentum components \vec{P} in the Taylor expandable region accessible to the “poor physicist”. The latter comes about because there are only the three non-trivial Pauli matrices that make the single particle energy vary in a linear way around the point of expansion. In this sense the number of spatial dimensions comes out as equal to the number of Pauli matrices.

9 Charge conservation

We should like to remark here that the existence of a dispersion relation for a type of particle giving the energy as a function of momentum, which we now assume to be conserved, can very easily lead to the effective conservation of the particle number for the “poor physicist”. The point is simply the following:

The poor physicist who has only very little energy at his disposal can normally only work with – i.e. produce or find manageable particles in – a very tiny part of the momentum range. If he seeks to go outside the very small range he will need bigger energy. By manageable we here mean that he can accelerate the particles in the momentum range in question, so that a filled Dirac sea is useless unless he can afford to bring the particles out of it. So only the part of the Dirac sea very close in energy to the Fermi surface is considered manageable. But now, generically, the tiny range of momentum with which he can work would, in a non-rotational invariant theory of

the type we consider here, almost certainly not just be for zero momentum. But we would rather expect some momentum range which is small in extension, compared to the fundamental scale of momentum, but surrounding a momentum value typically of order unity in the supposed very large fundamental scale unit.

This really means that all the particles, with which the poor physicist can work, have one huge momentum of order the fundamental scale plus a tiny little bit identified as the momentum he talks about in the laboratory. But this situation now has the consequence that in practice the particle number cannot change! Such a change would require that the big momentum of fundamental scale should be dispensed with in order to, say, destroy the particle. This will normally be very difficult, even with some violation of momentum conservation, since it would have to be a very special amount of momentum that should be dispensed with and there would be very little phase space for that to happen. With momentum conservation assumed, it becomes of course impossible unless the poor physicist can afford to build up to the huge momentum; he cannot do so in practice if, say, the huge momentum is of order the Planck momentum and he needs to transfer it onto a single or very few particles.

We can therefore say that our type of model gives a natural mechanism for generating a conserved particle number charge.

Avoiding particle anti-particle level repulsion

That there are such charge conservation mechanisms for the particles, which finally become relativistic particles, is actually quite needed for the following reason having to do with the CPT-theorem:

Both for the Weyl fermion and for the photon we have, due to CPT symmetry in a final relativistic theory, an exact degeneracy between a particle and an anti-particle having the same momentum. Suppose now the theory were quite generic and everything not forbidden should occur as terms in the Hamiltonian or commutators, and we had no selection rules like those arising from rotation symmetry at first. Then the particle and the anti-particle could mix and there would generically be a level repulsion, driving the energies of the two states with the same momentum away from each other! But that would be catastrophic for reproducing a theory satisfying the CPT-theorem.

However, as explained above, we indeed do have a charge conservation in our model for both the complex $\vec{\phi}(\vec{x})$ field and the Weyl field. In the boson case, it turns out to be the duality charge corresponding to the generator of duality transformations of the electromagnetic fields:

$$\vec{B} \longrightarrow \vec{B} \cos \theta + \vec{E} \sin \theta, \quad \vec{E} \longrightarrow \vec{E} \cos \theta - \vec{B} \sin \theta. \quad (92)$$

So we have a chance, at least, of obtaining totally degenerate CPT-connected states and thus a chance to get the final theory of relativity out. According to the potential problems associated with level repulsion, we should not have been able to obtain Majorana particles or bosons not having a charge or particle number. Otherwise we would have expected to generically obtain a CPT non-conserving theory.

So we can say that if one should find Majorana particles, except if they are due to some tiny correction starting from particles with a conserved particle number, it would be in contradiction with our present random dynamics model. Especially supersymmetry models, with lots of Majorana fermions, look a bit dangerous from the point of view of our random dynamics model.

10 Interaction of Weyl and photon fields

We shall now argue that it is indeed quite natural to obtain the well-known quantum electrodynamics, when we allow for interactions. It is a well-known problem for electrodynamics that there

is essentially no nice way to write down the interaction Lagrangian term – usually taken to be of the form $A_a \psi^\dagger \sigma^a \psi$ – without the use of the electromagnetic *four potential* A_a or A_μ . One can in principle, in some gauge, reconstruct the A_μ -field from an integral over the \vec{E} and \vec{B} fields. Such constructions will turn out to be non-local (in the sense that you integrate over different space-time regions) and rather inelegant. However above, we have managed to obtain the free Maxwell equations without using the A_μ field at any stage.

Let us recall that we introduced a field A_a in equation (87). That there should be such a field A_a is easily seen, by considering the most general linear equation of motion for a fermion with two components. That is to say we shall look at a Weyl-like equation or Schrödinger-like equation with two components:

$$i \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = - \sum_a \sigma^a i D_a \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (93)$$

This form of the equation will be the most general Hermitian Hamiltonian Schrödinger equation, if $i D_a$ are four general Hermitian operators in the geometrical, i.e. $\vec{x} - \vec{p}$, degrees of freedom. As above, we shall take the philosophy that, for the poor physicist, the momentum range can only be small compared to the fundamental scale, while \vec{x} may be large. Thus we Taylor expand $i D_a$ in \vec{p} , while each term has an arbitrary \vec{x} dependence:

$$-i D_a = A_a(\vec{x}) + \vec{p} \cdot \vec{V}_a(\vec{x}). \quad (94)$$

The notation here is suggestive of the physical interpretation we are suggesting. The dominant term in the Taylor expansion $A_a(\vec{x})$ is to be interpreted as the four-potential. However the index being a suggests that it is a so-called flat index, which means that it is associated with a vierbein, V_a^μ , formulation of general relativity, rather than the coordinate or Einstein index μ on A_μ , i.e. really $A_a = V_a^\mu A_\mu$. The next term $\vec{p} \cdot \vec{V}_a(\vec{x})$ is essentially just the vierbein we have to accept as coming out of the “random” momentum dependence of the fermion Hamiltonian around the two state degenerate point as we considered it above. Actually the Weyl field related vierbein V_a^μ is the extension of V_a^i in equation (94), by the further components $V_a^0 = \delta_a^0$ representing the left hand side of equation (93). This is discussed further below. Note that *a priori* the metric associated with the Maxwell field is different to the Weyl metric and, thus, this vierbein V_a^μ will not match with the Maxwell metric.

By second quantisation of the above single fermion Hamiltonian, we obtain the following second quantised Hamiltonian

$$\int \psi^\dagger (H_F + H_{INT}) \psi \, d^d \vec{x} = \int \psi^\dagger (V_a^\mu p_\mu + A_a) \sigma^a \psi \, d^d \vec{x}. \quad (95)$$

So we can say that we have argued for an interaction term

$$H_{INT} = \int A_a \psi^\dagger \sigma^a \psi \, d^d \vec{x} \quad (96)$$

as natural in \vec{x} -representation.

10.1 The interacting model

The philosophy is that we simply add this interaction term to the free particle approximation boson and fermion Hamiltonians. But now we remember that, for the bosons, we have relegated the complications into the commutators. It is therefore the commutators of the $\vec{\phi}$ -fields and the A_a -fields which may contain such possible complicated dynamics. It is also only this commutator which can identify or create a “same position” relation between the space in which the Weyl

fermion exists and the one in which the Maxwell equation photon exists. Indeed it is good to have in mind that, in the free approximation used up to now, there is absolutely no sense in saying that a boson and a Weyl fermion are at the same place. They live so to speak in completely different geometrical spaces. Even when we developed the interaction term (96), the position variable in the field $A_a(\vec{x})$ (later to be the electromagnetic 4-potential) is a position in the Weyl fermion geometrical space and we cannot yet identify this position variable with that for the Maxwell photon. Thus the identification of these two totally different geometrical spaces comes about only via the *commutators* of the A_a and the $\vec{\phi}$ fields. In our “perturbative” approach of considering the dominant terms in the sense of having fewest field operator factors, we approximate the commutator of $\vec{\phi}$ and A_a as a c-number times a δ -function

$$[\phi_i(\vec{x}), A_a(\vec{x}')] = F_{ia}(\vec{x})\delta(\vec{x} - \vec{x}'). \quad (97)$$

Regarding the background fields $F_{ia}(\vec{x})$ and $\vec{V}_a(\vec{x})$, we may take either of the two following attitudes

1. They are gravitational fields and depend on the dynamical “rest” variables.
2. They are background fields which act like coupling constants.

In the first case, we might hope one day to incorporate gravity into our model and we could then treat them as constant on the grounds of the gravitational field being weak and, hence, approximately taking the trivial form $g^{\mu\nu} = \eta^{\mu\nu}$ and $V_a^\mu = \delta_a^\mu$. In the second case, we could assume them to be constant on the grounds of simply putting translational invariance into our model *ab initio*.

We shall see now that, basically from the consistency of the equations of motion, we can find the form of the constant F_{ia} in this commutator up to a constant factor e being the electric charge of the fermion. In fact the equation of motion for the Maxwell fields reads

$$\dot{\phi}_i = i[H, \phi_i] = i[H_B + H_{INT}, \phi_i] = i\epsilon_{ikl}\frac{\partial}{\partial x_k}\phi_l(\vec{x}) - iF_{ia}\psi^\dagger(\vec{x})\sigma^a\psi(\vec{x}). \quad (98)$$

Here we used the notation of the coordinates associated with the metric for the Maxwell fields, so that the free Maxwell equations are without explicit vierbeins or metrics. In this formulation, however, the Weyl particle then needs a vierbein and the expression for the conservation of the Weyl fermion number becomes

$$\partial_\mu j^\mu = \partial_\mu (V_a^\mu j^a) \quad (99)$$

where

$$j^a = \psi^\dagger \sigma^a \psi \quad (100)$$

and V_a^μ is the constant vierbein.

10.2 Constraining the commutator coefficients F_{ia} by consistency of equations of motion

By taking the divergence of equation (98), it follows that

$$\frac{d}{dt}\text{div}\vec{\phi}(\vec{x}) = -i\partial_i (F_{ia}\psi^\dagger(\vec{x})\sigma^a\psi(\vec{x})), \quad (101)$$

showing that the time derivative of the divergence of $\text{div}\vec{\phi}$ only depends on the matter or fermion fields. If we imagine that this $\text{div}\vec{\phi}$ can be expressed as a function of all the fields of the theory, we may expand the time derivative

$$\text{div}\dot{\vec{\phi}} = \frac{\partial\text{div}\vec{\phi}}{\partial\vec{\phi}}\dot{\vec{\phi}} + \frac{\partial\text{div}\vec{\phi}}{\partial\psi_i}\dot{\psi}_i \quad (102)$$

by differentiating with respect to the various fields. Since, according to the equation of motion for ψ , even $\dot{\psi}$ at first only contains A_a and ψ but no $\vec{\phi}$ field, it is indicated that $\text{div } \vec{\phi}$ should not depend on the ϕ_i 's but rather on $\dot{\psi}$ and perhaps on A_a . Ignoring possible dependence on A_a , we want the time derivative of $\text{div } \vec{\phi}$ to be a quadratic expression in ψ . Then, since the equation of motion leaves $\dot{\psi}$ linear in ψ , we really need $\text{div } \phi$ to be a quadratic expression in ψ .

We are therefore driven to make an ansatz of the form

$$\text{div } \vec{\phi} = \psi^\dagger \mathcal{O} \psi \quad (103)$$

where to keep fermion number conservation, we should have the form with one ψ^\dagger and one ψ . We expand the operator \mathcal{O} , which is constant (in space), as

$$\mathcal{O} = W_a \sigma^a. \quad (104)$$

Note that, since $\text{div } \phi(\vec{x})$ is not Hermitian in \vec{x} -representation, it is possible that the four numbers W_a are complex.

Now let us use the equation of motion for the fermion fields (93), so as to compute the time derivative of our ansatz (103)

$$\frac{d}{dt} \text{div } \vec{\phi} = -\frac{\partial \psi^\dagger}{\partial \vec{x}} \vec{V}_a \sigma^a W_b \sigma^b \psi - \psi^\dagger W_b \sigma^b \vec{V}_a \sigma^a \frac{\partial \psi}{\partial \vec{x}}, \quad (105)$$

where we used the Hermitian conjugate of equation (93) without the A_a term

$$i \frac{d}{dt} \psi^\dagger = -i \frac{\partial \psi^\dagger}{\partial \vec{x}} V_a \sigma^a. \quad (106)$$

By rearrangement of the terms in equation (105), we further obtain

$$\frac{d}{dt} \text{div } \vec{\phi} = -\frac{\partial}{\partial \vec{x}} \left(\psi^\dagger \frac{1}{2} [\vec{V}_a \sigma^a, W_b \sigma^b]_+ \psi \right) - \psi^\dagger \frac{1}{2} [W_b \sigma^b, \vec{V}_a \sigma^a] \frac{\overleftrightarrow{\partial}}{\partial \vec{x}} \psi \quad (107)$$

According to equation (101) we do not want a term of the second type with the derivative $\frac{\overleftrightarrow{\partial}}{\partial \vec{x}}$ acting both ways, but only a term of the first type, where the $\frac{\partial}{\partial \vec{x}}$ differentiation acts outside the $\psi^\dagger \dots \psi$ product. The requirement that the second term disappears is that for all $i = 1, 2, 3$ we have

$$W_b V_a^i \epsilon^{bac} = 0 \quad (108)$$

So in the case – normally true – that the 3×3 matrix $(V_a^i)_{a=1,2,3}$ is non-degenerate, we conclude

$$W_b = 0 \quad \text{for } b = 1, 2, 3. \quad (109)$$

In order that the first term in equation (107) should agree with equation (101) we need

$$\sigma^a F_{ia} = \frac{1}{2i} [V_a^i \sigma^a, W_b \sigma^b]_+ \quad (110)$$

which in turn gives

$$i F_{i0} = \sum_{b=0}^3 V_b^i W_b = W_0 V_0^i \quad (111)$$

and

$$i F_{ia} = W_0 V_a^i \quad \text{for } a \neq 0. \quad (112)$$

But this means that we only get consistent equations when F_{ia} takes this special form. Notice that, in our Hamiltonian (or absolute time) presentation of the fermion theory, we did not use a full 16 component vierbein, but only the 12 components with the Einstein index being $i = 1, 2, 3$. We did not need the components V_a^0 . Although we already introduced the further components in the first part of this section, we shall formally re-introduce them here. We do that by re-writing the expression (103) as

$$\text{div } \vec{\phi} = W_0 \psi^\dagger V_a^0 \sigma^a \psi \quad (113)$$

where W_0 can be considered a complex charge, as we also do for the following expression:

$$iF_{ia} \psi^\dagger \sigma^a \psi = W_0 \psi^\dagger V_a^i \sigma^a \psi. \quad (114)$$

Here we used equations (111) and (112). Consistency of equation (113) with equations (103) and (104), requires

$$W_a = W_0 V_a^0. \quad (115)$$

Combined with equation (109) this of course gives

$$V_0^0 = 1, \quad V_b^0 = 0, \quad \text{for } b \neq 0. \quad (116)$$

Since our Weyl equation of motion was simply (93), we see that it can indeed be precisely expressed by means of a four dimensional vierbein of just the form to which we arrived, namely (116) and the already needed 12 components \vec{V}_a in, say, equation (94). In fact the equation of motion (93) takes the following four dimensional form

$$V_a^\mu \sigma^a iD_\mu \psi = 0 \quad (117)$$

where now

$$D_\mu = (V^{-1})_\mu^a \left(D_a + \delta_a^0 \frac{d}{dt} \right) = (V^{-1})_\mu^a \left(iA_a + ip_i V_a^i + \delta_a^0 \frac{d}{dt} \right) = iA_\mu + \frac{\partial}{\partial x^\mu} \quad (118)$$

and A_μ is defined so that

$$A_a = V_a^\mu A_\mu. \quad (119)$$

From such a vierbein form of the Weyl equation, one immediately finds that the conserved 4-current is

$$j^\mu = V_a^\mu \psi^\dagger \sigma^a \psi \quad (120)$$

obeying

$$\partial_\mu j^\mu = 0. \quad (121)$$

From equations (111) and (112), we have

$$iF_{ia} = W_0 V_a^i. \quad (122)$$

So the equation of motion (98) becomes

$$\dot{\phi}_i(\vec{x}) = i\epsilon_{ikl} \frac{\partial}{\partial x_k} \phi_l(\vec{x}) - W_0 j^i(\vec{x}) \quad (123)$$

where j^i is one of the spatial components of (120). We must admit that it looks strange here to have put an upper index on j^i ; it is a consequence of our notation for the vierbein V_a^μ with an upper Einstein index μ . Using the space-time formulation (83), this equation can be written as the spatial components for the covariant-looking equation

$$\partial_\alpha (F^{\alpha\beta} - i\mathcal{F}^{\alpha\beta}) = -iW_0 j^\beta, \quad (124)$$

i.e. we got it for $\beta = i$, $i = 1, 2, 3$. For $\beta = 0$ equation (124) corresponds to equation (103). Taking the Hermitian conjugate of this equation gives us

$$\partial_\alpha (F^{\alpha\beta} + i\mathcal{F}^{\alpha\beta}) = iW_0^* j^\beta, \quad (125)$$

since j^β , $F^{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta}$ are all Hermitian operators.

Adding and subtracting, we obtain

$$\partial_\alpha F^{\alpha\beta} = \frac{1}{2} (iW_0^* - iW_0) j^\beta \quad (126)$$

and

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = \frac{1}{2} (W_0 + W_0^*) j^\beta \quad (127)$$

which shows that the Weyl fermion has come to function as a dyon with electric e and magnetic g charges given by

$$e = \text{Im } W_0, \quad g = \text{Re } W_0. \quad (128)$$

Although these equations look completely Lorentz covariant, we still actually have a Lorentz non-invariant theory, because the current $j^\beta(\vec{x})$ is given by (120) and contains the vierbein V_a^β which is constant in space and time. We can make this very important flaw manifest, by writing equations (126) and (127) with the vierbein explicitly exposed:

$$\partial_\alpha F^{\alpha\beta}(\vec{x}) = e V_a^\beta \psi^\dagger(\vec{x}) \sigma^a \psi(\vec{x}), \quad \partial_\alpha \mathcal{F}^{\alpha\beta} = g V_a^\beta \psi^\dagger(\vec{x}) \sigma^a \psi(\vec{x}). \quad (129)$$

Thus, even with our optimistic speculations, we did not obtain full Lorentz invariance at first.

11 Transforming away a dyon

As already mentioned in section 7, the extraction of \vec{E} and \vec{B} from $\vec{\phi}$ can be made in different conventions, provided one does not require that the condition $\vec{\phi}(\vec{x}) = \vec{\phi}^\dagger(-\vec{x})$ from the Hermiticity of the momentum representation field be just represented as invariance of the field *configuration* under P and PT. Liberated from such requirements we may use equation (66), which has an extra constant phase factor $e^{i\delta}$ put on $\vec{\phi}(\vec{x})$, in order to extract \vec{E} and \vec{B} .

With this extraction of $\vec{B} + i\vec{E}$ we get

$$\partial_\alpha (e^{-i\delta} (F^{\alpha\beta} - i\mathcal{F}^{\alpha\beta})) = -iW_0 V_a^\beta j^a \quad (130)$$

and

$$\partial_\alpha (e^{i\delta} (F^{\alpha\beta} + i\mathcal{F}^{\alpha\beta})) = iW_0^* V_a^\beta j^a \quad (131)$$

instead of equations (124) and (125). These revised equations are of the usual form (129) but now with

$$e = \text{Im} (W_0 e^{i\delta}), \quad g = \text{Re} (W_0 e^{i\delta}). \quad (132)$$

With this phase δ available as a choice of interpretation, we can arrange *for a single fermion field* that, for example, the magnetic charge g is zero. In this way we can achieve conventional quantum electrodynamics *except that we still have the non-trivial vierbein*. This means that the “electron” has its own metric tensor, which is different from that of the Maxwell fields.

12 The orbifold nature of the Universe: an extra prediction

In [2], where we assumed translational invariance at the outset, the space on which the Maxwell fields were defined was indeed an orbifold. The orbifold identification in this case corresponds to identifying the positions at \vec{x} and at $-\vec{x}$. This simple orbifold identification corresponds to the reflection symmetry relations in phase space (orbifolding of phase space) discussed in section 4 of this paper. However, we have replaced the Weyl fields by effectively complex fields and we do not have a corresponding symmetric state (of the fermion fields). Therefore we effectively get the fermion fields *doubled*, when we live on the orbifold. Suppose the Weyl fields, which are genuinely near \vec{x} , are left-handed. Then it is not difficult to see that the fields (also left-handed of course) near $-\vec{x}$ will appear as right-handed to the physicist at \vec{x} . Thus the effect of this reflection symmetry or orbifold structure is really that the Weyl fermions get doubled to become Dirac fermions. This poses a phenomenological problem for our model, in as far as weak interaction charges are not symmetric under left going to right. So such a vectorlike symmetry prediction from the orbifold feature will prevent an extension of our model to include the weak interactions, unless we can somehow provide a crutch to cure this problem.

With this reflection identification $\vec{\phi}(\vec{x}) = \vec{\phi}^\dagger(-\vec{x})$ we do not have locality, as was tacitly assumed in the derivation of the Maxwell equations (124) and (125), and thus these equations are strictly speaking wrong. Rather we must consider together the contributions to the Hamiltonian around a point \vec{x} and the reflected point $-\vec{x}$. Effectively we may “bring” the interaction of the Maxwell field with the Weyl fermion field at $-\vec{x}$ over to the point \vec{x} , by means of a reflection symmetry operation P or PT. Really it is easy to see that we can describe the effect of the interaction of the $\vec{\phi}(-x)$ field with the *a priori* left Weyl field – in the region around $-\vec{x}$ – as an interaction of the field near \vec{x} , i.e. $\vec{\phi}(\vec{x})$, with a right-handed partner field, giving simply an extra term on the right-hand side of the Maxwell equations.

The degrees of freedom of the fermion near $-\vec{x}$ can, in practice, be considered quite independently of those near \vec{x} and thus are best described by completely separate fermionic fields, marked say by an index R to remind us that they now represent a right-handed Weyl particle. The ones at \vec{x} could then be marked analogously by an index L . The vierbein V_{Ra}^μ for the ψ_R degrees of freedom are obtained by the parity operation on the original vierbein for the *a priori* left Weyl field V_a^μ . That is to say

$$V_{Ra}^\mu = -(-1)^{\delta_a^\mu} V_a^\mu, \quad V_{La}^\mu = V_a^\mu. \quad (133)$$

The full Maxwell equations for the doubled theory become

$$\partial_\alpha F^{\alpha\beta} = e V_{La}^\beta \psi_L^\dagger \sigma^a \psi_L + e V_{Ra}^\beta \psi_R^\dagger \sigma^a \psi_R \quad (134)$$

and

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = g V_{La}^\beta \psi_L^\dagger \sigma^a \psi_L + g V_{Ra}^\beta \psi_R^\dagger \sigma^a \psi_R. \quad (135)$$

This means that the Weyl fermion components R and L *a priori* each have their own vierbein and thereby also their own metric tensor. However, when we remember that due to the absolute time we have $V_i^0 = 0$ from equation (116), we see that with

$$g_R^{\mu\nu} = \eta^{ab} V_{Ra}^\mu V_{Rb}^\nu, \quad g_L^{\mu\nu} = \eta^{ab} V_{La}^\mu V_{Lb}^\nu, \quad (136)$$

we have in fact

$$g_L^{\mu\nu} = g_R^{\mu\nu}, \quad (137)$$

because the sign from the parity reflection is squared in the metrics. Thus it actually turns out that the whole Dirac fermion – here massless – behaves in a Lorentz invariant way by itself. However the electromagnetic fields have a different metric.

13 Conclusion and outlook

We started from a very general — essentially random — quantum field theory, with the second quantized field operators only defined as functions of general coordinates \vec{x}_{pre} . The interpretation of these general coordinates, in terms of physical position and momentum, was then made in a way depending on the Hamiltonian and commutator functions. This required a drastic transformation of the single particle degrees of freedom, in order to reach the identification with known physics — meaning quantum electrodynamics. The procedure was based on Taylor expanding in the low energy and corresponding deviations in what we interpreted as momentum, so that it only works in the low energy limit relevant for the so-called “poor physicist”. We obtained the 3+1 dimensional Maxwell equations and Weyl equations in the free approximation, as already done in an earlier article [2] under the assumption of translational invariance.

Furthermore we generalised the model to include a general interaction between the fermions and the Maxwell fields. We also made an attempt to even derive translational invariance similarly. The latter attempt was less successful, although there is some hope to succeed by including gravity.

13.1 The great speculation that gravity may solve our remaining problems!

We did not really succeed fully in deriving quantum electrodynamics from a completely random or complicated theory in the low energy limit. But we actually came so far that there is a hope of reaching this goal, if somehow the gravity fields of general relativity emerge from our approach (or are simply added by assumption) and the background fields $\sqrt{g^{ij}_{Maxwell}}$ and V_a^μ are identified with the gravitational fields.

By speculatively incorporating gravity into the model in this way, we may be able to resolve some of the major problems still remaining in our model:

a) The problem of there still *not being translational invariance* would be solved by the background fields depending on the position, $\sqrt{g^{ij}_{Maxwell}}$ and V_a^μ , now being interpreted as gravitational fields. Then the lack of translational invariance would just be interpreted as the effect of gravitational fields being present, which are not translational invariant. Thus the question of why we have translational invariance would be brought back to the question of why the gravitational fields have arranged themselves in such a way that the space-time is so flat as to have translational invariance approximately.

b) The rudiments of the *extra dimensions* must not disturb our results, in order that our model be truly viable. The three spatial dimensions and the one time dimension are singled out by the velocity of the particles being of order unity in these directions, while the velocity in the extra directions relative to the 3 spatial dimensions is zero. If we were optimistic then we could assume that the gravity theory would, in an analogous way, single out three dimensions as the only ones in which rapid particle motion could take place. This would mean that the gravitational Lagrangian would lack the gradient squared terms corresponding to the gradients in the *extra* directions. In order to see what this could mean, we may think of the analogous thing happening for an ordinary gauge field:

In [5] we studied a lattice gauge theory in a phase in which there was, crudely speaking, confinement in some directions — dimensions — while having an ordinary electrodynamics-like phase in say four dimensions. This was achieved by having a strong lattice gauge coupling — meaning a small coefficient on the lattice action plaquette terms — in the extra dimensions, while having a weaker one in the normal dimensions. In this phase a gauge field charge could not propagate from one “layer” to the next, meaning it could not propagate in the dimensions with strong coupling behaviour but would be confined to its layer. In the suggested analogy, the analogue of the charge would be the energy and momentum in gravity theory. Then having zero velocity particle

motions in the more than $3 + 1$ dimensions, which means zero gradient term coefficient in these directions, is in turn analogous to the strong coupling plaquette, i.e. the plaquette with a small coefficient on it. So we would expect a strong coupling behaviour with momentum and energy confined in the more than $3 + 1$ dimensions. This would mean that there would be layers and no momentum or energy could go from layer to layer. In practice this essentially means that one layer cannot communicate with the next, at least not in a momentum transferring way. That would presumably mean that the communication between layers would only be of the form that one layer could influence effective couplings on neighbouring layers. However no special communication to particular points in one layer should be possible from the other layers!

13.2 What is so special about the 3 space dimensions ?

A very remarkable prediction from our models is that the number of spatial dimensions, in which the speed of motion will be appreciable, is just *three*, in wonderful agreement with the experimental fact that we have just three spatial dimensions. It should be stressed that we even get this result twice, namely both for the boson and the fermion cases! Is there a simple reason why we should get the number three [6]-[12] each time? Well, we should think of what would have happened if we should have got a higher dimension, bearing in mind that we have *not* assumed rotational invariance and have effectively used a generic Hamiltonian. It follows that there should thus be great difficulty in obtaining degenerate levels because of level repulsion.

In theories with Lorentz (including rotational) invariance there is, of course, no need for any fine-tuning to have the different spin (or helicity for massless particles) states be degenerate for a given momentum. This is because they have different values for the angular momentum component, which in such theories is a conserved quantity. But now in our model we do *not* put angular momentum in as an *a priori* conserved quantity. Thus we should not expect that states only distinguished by their – for us *a priori* non-conserved – angular momentum component to be degenerate.

Now the little group for a massless particle – and we have seen that the particles predicted in our model are at least at first massless – in $(D - 1) + 1$ dimensions is $SO(D - 2)$ extended by the inclusion of some non-compact Lorentz transformations. The extensions mentioned are not used for separating spin or helicity eigenstates and we shall ignore them here. Thus the important part $SO(D - 2)$ of the little group for the massless particle is non-abelian for all space-time dimensions $D > 4$. Only for the spatial dimension $d = D - 1 = 3$ or smaller will a non-trivial spin not automatically imply degeneracy of the different states. The point is that non-trivial representations of $SO(D - 2)$ for $D - 1 > 3$ necessarily have representation dimensions bigger than unity, which means degeneracy. If rotation is not a good conserved symmetry, such a degeneracy should never occur. Thus, for $D > 4$, we should not be successful with a model like ours in reproducing Lorentz invariance, except for scalars which anyway will lack the mass protection needed to make them accessible to the “poor physicist”. Therefore we could not possibly have got more than the experimental number of space dimensions $D - 1 = 3$. Actually, as we discussed in section 9, the CPT-theorem poses a problem at first even in $D - 1 = 3$ dimensions: The anti-particle of say the Weyl particle which we did obtain is in fact degenerate by CPT symmetry to the original particle. By a similar argument, such a degeneracy would be expected to be impossible in our model. This is indeed true but, due to an effective charge conservation, the two states can be distinguished by an effectively conserved quantity, i.e. the charge, and cannot mix. Were it not for this charge, the particle should have become a very heavy Majorana particle not accessible to the poor physicist and the degeneracy would have disappeared also.

Even in the photon case, there must have been an effective charge for otherwise the level repulsion would have made it impossible to get the usual CPT-invariant picture out. In this case, it is a duality transformation generating charge that does the job.

There is a natural mechanism for these abelian symmetries to arise accidentally, due to particles carrying a large fundamental scale unit of momentum in addition to the physical momentum observable by the poor physicist. However it may be much harder, in such an accidental way, to obtain non-abelian “rotation” symmetries that could prevent the level repulsion between different components of the little group representation. Such non-abelian symmetries would be needed for the case of higher than $3 + 1$ dimensions, since then the rotation group about the direction of motion of a particle would be non-abelian. In this way we understand why the number of spatial dimension predicted by us is at most three.

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